Measuring conflict between possibilistic uncertain information through belief function theory

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Abstract. Dempster Shafer theory of evidence (DS theory) and possibility theory are two main formalisms in modelling and reasoning with uncertain information. These two theories are inter-related as already observed and discussed in many papers (e.g. [DP82,DP88b]). One aspect that is common to the two theories is how to quantitatively measure the degree of conflict (or inconsistency) between pieces of uncertain information. In DS theory, traditionally this is judged by the combined mass value assigned to the emptyset. Recently, two new approaches to measuring the conflict among belief functions are proposed in [JGB01,Liu06]. The former provides a distance-based method to quantify how close a pair of beliefs is while the latter deploys a pair of values to reveal the degree of conflict of two belief functions. On the other hand, in possibility theory, this is done through measuring the degree of inconsistency of merged information. However, this measure is not sufficient when pairs of uncertain information have the same degree of inconsistency. At present, there are no other alternatives that can further differentiate them, except an initiative based on coherence-intervals ([HL05a,HL05b]). In this paper, we investigate how the two new approaches developed in DS theory can be used to measure the conflict among possibilistic uncertain information. We also examine how the reliability of a source can be assessed in order to weaken a source when a conflict arises.

1 Introduction

Pieces of uncertain information that come from different sources often do not agree with each other completely. There can be many reasons for this, such as, inaccuracy in sensor data reading, nature errors occurred in experiments, reliabilities of sources, etc. When inconsistent information needs to be merged, assessing the degree of conflict among information plays a crucial role in deciding which combination mode would be best suited [DP94].

In possibility theory, the well established method is to measure the degree of inconsistency between two pieces of uncertain information. This measure is not enough when multiple pairs of uncertain information have the same degree of inconsistency. We need to further identify subsets of sources that contain information more "close" to each other. Currently, there are no approaches to fulfilling this objective, except a coherence-interval based scenario proposed in [HL05a,HL05b]. More robust methods are needed to measure the conflict among pieces of information more effectively.

Two fundamental functions defined in possibility theory are possibility measures and necessity measures. In the context of Dempster-Shafer theory of evidence (DS theory for short), these two measures are special cases of plausibility and belief functions. Naturally, DS theory faces the same question as how conflict should be measured among belief functions. Recently, two different approaches were proposed to quantitatively judge how conflict a pair of uncertain information is [JGB01,Liu06]. One approach calculates the distance between two belief functions and another evaluates a pair of values consisting of the difference between betting commitments and a combined mass assigned to the emptyset. Both methods provide a better measure about the conflict among belief functions then the traditionally used approach in DS theory, that is, the use of the mass value assigned to the emptyset after combination.

In this paper, we take the advantage that possibility and necessity measures are special cases of plausibility and belief functions and investigate the effect of applying the two new approaches introduced above in DS theory to possibilistic uncertain information. Properties and potential applications of this investigation are explored too. In addition, we look at the issues of assessing the reliability of sources to assist resolving conflict through weakening the opinion from less reliable sources.

We will proceed as follows: in Section 2, we review the basics in possibility theory and DS theory. In Section 3, we present the relationships and properties between the two theories. In Section 4, we investigate how the approaches for inconsistent assessment in DS theory can be applied to possibilistic uncertain information. In Section 5, we examine how individual agent's judgement can be assessed, in order to discount or discarded some sources in a highly conflict situation. Finally in Section 6, we summarize the main contributions of the paper.

2 Brief review of DS theory and possibility theory

2.1 Basics of Dempster-Shafer theory

Let Ω be a finite set containing mutually exclusive and exhaustive solutions to a question. Ω is called the *frame of discernment*.

A basic belief assignment (bba) [Sme04] is a mapping $m : 2^{\Omega} \to [0,1]$ that satisfies $\sum_{A\subseteq\Omega} m(A) = 1$. In Shafer's original definition which he called *the basic probability assignment* [Sha76], condition $m(\emptyset) = 0$ is required. Recently, some of the papers on Dempster-Shafer theory, especially since the establishment of the Transferable Belief Model (TBM) [SK94], condition $m(\emptyset) = 0$ is often omitted. A bba with $m(\emptyset) = 0$ is called a normalized bba and is known as a *mass function*.

$m(A)$ defines the amount of belief to the subset A exactly, not including any subsets in A. The total belief in a subset A is the sum of all the mass assigned to all subsets of A. This function is known as *a belief function* and is defined as $Bel : 2^{\Omega} \to [0,1]$.

$$Bel(A) = \Sigma_{B\subseteq A} m(B)$$

When $m(A) > 0$, A is referred to as a *focal element* of the belief function.

A *plausibility function*, denoted Pl, is defined as follows, where $Pl : 2^{\Omega} \to [0,1]$.

$$Pl(A) = 1 - Bel(\bar{A}) = \Sigma_{B\cap A\neq\emptyset}\, m(B)$$

where $\bar{A}$ is the complementary set of A.

Two pieces of evidence expressed in bbas from distinct sources are usually combined using Dempster's combination rule. The rule is stated as follows.

Definition 1. *Let m_1 and m_2 be two bbas, and let $m_1 \oplus m_2$ be the combined bba.*

$$m_1 \oplus m_2(C) = \frac{\Sigma_{A \cap B = C}\ (m_1(A) \times m_2(B))}{1 - \Sigma_{A \cap B = \emptyset}\ (m_1(A) \times m_2(B))}$$

When $m_1 \oplus m_2(\emptyset) = \Sigma_{A \cap B = \emptyset}\ (m_1(A) \times m_2(B)) = 1$, the two pieces of evidence are totally contradict with each other and cannot be combined with the rule.

Definition 2. *[Sme04] Let m be a bba on Ω. Its associated pignistic probability function $BetP_m : \Omega \to [0,1]$ is defined as*

$$BetP_m(\omega) = \sum_{A \subseteq \Omega, \omega \in A} \frac{1}{|A|} \frac{m(A)}{1 - m(\emptyset)}, \quad m(\emptyset) \neq 1 \tag{1}$$

where $|A|$ is the cardinality of subset A.

The transformation from m to $BetP_m$ is called the *pignistic transformation*. When an initial bba gives $m(\emptyset) = 0$, $\frac{m(A)}{1-m(\emptyset)}$ is reduced to $m(A)$. Value $BetP_m(A)$ is referred to as the *betting commitment to* A.

2.2 Possibility theory

Possibility theory is another popular choice for representing uncertain information ([DP88a,BDP97], etc). At the semantic level, a basic function in possibility theory is a *possibility distribution* denoted as π which assigns each possible world in the frame of discernment Ω a value in $[0,1]$ (or a set of graded values).

From a possibility distribution, two measures are derived, a possibility measure (demoted as Π) and a necessity measure (denoted as N). The former estimates to what extent the true event is believed to be in the subset and the latter evaluates the degree of necessity that the subset is true. The relationships between π, Π and N are as follows.

$$\Pi(A) = \mathsf{max}(\{\pi(\omega) | \omega \in A\}) \text{ and } N(A) = 1 - \Pi(\bar{A}) \tag{2}$$

$$\Pi(2^{\Omega}) = 1 \text{ and } \Pi(\emptyset) = 0 \tag{3}$$

$$\Pi(A \cup B) = \mathsf{max}(\Pi(A), \Pi(B)) \text{ and } N(A \cap B) = \mathsf{min}(N(A), N(B)) \tag{4}$$

The usual condition associated with π is that there exists $\omega_0 \in \Omega$ such that $\pi(\omega_0) = 1$, and in which case π is said to be normal. It is not always possible to obtain a possibility distribution from a piece of evidence. Most of the time, uncertain information is expressed as a set of weighted subsets (or a set of weighted formulas in possibilistic logic). A weighted subset (A, α) is interpreted as that the necessity degree of A is at least to α, that is, $N(A) \geq \alpha$.

A piece of possibilistic uncertain information usually specifies a partial necessity measure. Let $\Omega = \{\omega_1, .., \omega_n\}$, and also let $A_i = \{\omega_{i_1}, .., \omega_{i_x}\}$ in order to make the subsequent description simpler. In this way, a set of weighted subsets constructed from a piece of uncertain information is defined as $\{(A_i, \alpha_i), i = 1, .., p\}$, where α_i is the lower bound on the degree of necessity $N(A_i)$. In the following, we call a set of weighted subsets a *possibilistic information base (PIB for short)* and denote such a base as K.

There is normally a family of possibility distributions associated with a given set of weighted subsets, with each of the distributions satisfying the condition

$$1 - \max\{\pi(\omega) | \omega \in \bar{A}_i\} \geq \alpha_i$$

which guarantees that $N(A_i) \geq \alpha_i$. Let $\{\pi_j, j = 1, .., m\}$ be all the possibility distributions that are compatible with $\{(A_i, \alpha_i), i = 1, .., p\}$. A possibility distribution $\pi_l \in \{\pi_j, j = 1, .., m\}$ is said to be the least specific possibility distribution among $\{\pi_j, j = 1, .., m\}$ if $\nexists \pi_t \in \{\pi_j, j = 1, .., m\}, \pi_t \neq \pi_l$ such that $\forall \omega, \pi_t(\omega) \geq \pi_l(\omega)$.

A common method to select one of the compatible possibility distributions is to use the *minimum specificity principle* [DP87] which allocates the greatest possibility degrees in agreement with the constraints $N(A_i) \geq \alpha_i$. This possibility distribution always exists and is defined as ([DP87,BDP97])

$$\forall \omega \in \Omega, \pi(\omega) = \begin{cases} \min\{1 - \alpha_i | \omega \notin A_i\} & \\ \quad = 1 - \max\{\alpha_i | \omega \notin A_i\} & \text{when } \exists A_i \text{ s. t. } \omega \notin A_i \\ 1 & \text{otherwise} \end{cases} \quad (5)$$

A possibility distribution is not normal if $\forall \omega, \pi(\omega) < 1$. The value $1 - \max_{\omega \in \Omega} \pi(\omega)$ is called *the degree of inconsistency* of the PIB and is denoted as $Inc(K)$. Given a PIB $\{(A_i, a_i),\ i = 1, .., p\}$, this PIB is *consistent* iff $\cap_i A_i \neq \emptyset$.

The two basic combination modes in possibility theory are the *conjunctive* and the *disjunctive* modes for merging possibility distributions ([BDP97]) when n possibility distributions are given on the same frame of discernment. For example, if we choose *min* and *max* as the conjunctive and disjunctive operators respectively, then

$$\forall \omega \in \Omega, \pi_{cm}(\omega) = \min_{i=1}^{n}(\pi_i(\omega)),\ \forall \omega \in \Omega, \pi_{dm}(\omega) = \max_{i=1}^{n}(\pi_i(\omega)) \quad (6)$$

A conjunction operator is used when it is believed that all sources are reliable and these sources agree with each other whilst a disjunctive operator is applied when it is believed that some sources are reliable but it is not known which of these sources are. A conjunction operator can lead to a new possibility distribution that is not normal when some sources are not in agreement, even though all the original possibility distributions are normal. When this happens, the merged possibility distribution expresses an inconsistency among the sources.

3 Belief functions verse necessity measures

In [Sha76], a belief function is called a *consonant function* if its focal elements are nested. That is, if S_1, S_2 ,.., S_n are the focal elements with S_{i+1} containing more elements than S_i, then $S_1 \subset S_2 \subset .. \subset S_n$. Let Bel be a consonant function, and Pl be its corresponding plausibility function, Bel and Pl have the following properties:

$$Bel(A \cap B) = \mathsf{min}(Bel(A), Bel(B)) \texttt{ for all } A, B \subseteq 2^{\Omega}.$$

$$Pl(A \cup B) = \mathsf{max}(Pl(A), Pl(B)) \texttt{ for all } A, B \subseteq 2^{\Omega}.$$

These two properties are exactly the requirements of necessity and possibility measures in possibility theory. Necessity and possibility measures are special cases of belief and plausibility functions.

Furthermore, a *contour function* $f : \Omega \rightarrow [0, 1]$, for a consonant function is defined through equation

$$f(\omega) = Pl(\{\omega\})$$

For a subset $A \subseteq \Omega$,

$$Pl(A) = \mathsf{max}_{\omega \in A} f(\omega) \tag{7}$$

Equation (7) matches the definition of possibility measure from a possibility distribution, so a contour function is a possibility distribution.

The procedure to derive a bba from a possibility distribution is stated below.

Proposition 1. *([HL06]) Let π be a possibility distribution on frame of discernment Ω and is normal. Let B_1, B_2,.., B_p and B_{p+1} be disjoint subsets of Ω such that $\pi(\omega_i) = \pi(\omega_j)$ when both $\omega_i, \omega_j \in B_i$; $\pi(\omega_i) > \pi(\omega_j)$ if $\omega_i \in B_i$ and $\omega_j \in B_{i+1}$; $\pi(\omega_i) = 0$ if $\omega_i \in B_{p+1}$ then the following properties hold:*

1. *Let $A_i = \cup\{B_j | j = 1, .., i\}$ for $i = 1, 2, .., p$, then subsets $A_1, A_2, .., A_p$ are nested;*
2. *Let $m(A_i) = \pi(\omega_i) - \pi(\omega_j)$ where $\omega_i \in B_i$ and $\omega_j \in B_{i+1}$ for $i = 1, .., p-1$. Let $m(A_p) = \pi(\omega)$ where $\omega \in B_p$. Then m is a bba on focal elements A_i;*
3. *Let Bel be the belief function corresponding to m defined above, then Bel is a consonant function.*

Subset B_1 (or focal element A_1) is called the *core* of possibility distribution π which contains the most plausible interpretations [BK01]. The nature of Proposition 1 was first observed in [DP82] where the relationship between the possibility theory and DS theory was discussed. This relationship was further referred to in several papers subsequently ([DP88b,DP98b,DNP00]).

Example 1 *Let π be a possibility distribution on $\Omega = \{\omega_1, ..., \omega_4\}$ where*

$$\pi(\omega_1) = 0.7, \pi(\omega_2) = 1.0, \pi(\omega_3) = 0.8, \pi(\omega_4) = 0.7$$

The disjoint subsets for π are

$$B_1 = \{\omega_2\}, \quad B_2 = \{\omega_3\}, \quad B_3 = \{\omega_1, \omega_4\}$$

and the corresponding focal elements as well as bba m are

$$A_1 = B_1, \qquad A_2 = B_1 \cup B_2, A_3 = B_1 \cup B_2 \cup B_3$$
$$m(A_1) = 0.2, m(A_2) = 0.1, \quad m(A_3) = 0.7$$

Proposition 2. *Let π be a possibility distribution on frame of discernment Ω and be normal. Let $BetP$ be the pignistic probabilistic function of the corresponding bba m derived from π. Then $BetP(\omega_i) \geq BetP(\omega_j)\ iff\ \pi(\omega_i) \geq \pi(\omega_j)$.*

Proof Let the collection of disjoint subsets satisfying conditions in Proposition 1 be $B_1, B_2, \ldots, B_{p+1}$ and let the set of focal elements be $A_1, A_2, \ldots, A_p$. Without losing generality, we assume $\omega_i \in B_1$ and $\omega_j \in B_2$, so $\pi(\omega_i) \geq \pi(\omega_j)$. Based on Equation 1,

$$BetP(\omega_i) = \frac{m(A_1)}{\mid A_1 \mid} + \frac{m(A_2)}{\mid A_2 \mid} + \ldots + \frac{m(A_p)}{\mid A_p \mid}$$

and

$$BetP(\omega_j) = \frac{m(A_2)}{\mid A_2 \mid} + \ldots + \frac{m(A_p)}{\mid A_p \mid}$$

It is obvious that $BetP(\omega_i) \geq BetP(\omega_j)$.

$\diamond$

In fact, if the elements in Ω are ordered in the way such that $\pi(\omega_1) \geq \pi(\omega_2) \geq ... \geq \pi(\omega_n)$, then the inequality $BetP(\omega_1) \geq BetP(\omega_2) \geq ... \geq BetP(\omega_n)$ holds. Proposition 2 is valid even when a possibility distribution is not normal. In that case, $m(\emptyset) = 1 - \pi(\omega | \omega \in B_1)$. This proposition says that the more plausible a possible world is, the more betting commitment it carries.

Proposition 3. *Let π_1 and π_2 be two possibility distributions on frame of discernment Ω for two PIBs and be normal. Let K be the conjunctively merged PIB. Assume m_1 and m_2 are the bbas derived from π_1 and π_2 respectively. Then the following properties hold.*

1. $Inc(K) = 0$ iff $m_1 \oplus m_2(\emptyset) = 0$
2. $Inc(K) = 1$ iff $m_1 \oplus m_2(\emptyset) = 1$
3. $Inc(K) > 0$ iff $m_1 \oplus m_2(\emptyset) > 0$

Proof We assume the conjunctive operator used in the proof is *min*. In fact, this proof is equally applicable to the other two commonly used conjunctive operators, namely, *product* and *linear product*.

Let B_{π_1} and B_{π_2} be the two cores for possibility distributions π_1 and π_2 respectively.

We first prove $Inc(K) = 0$ iff $m_1 \oplus m_2(\emptyset) = 0$. When $Inc(K) = 0$, the conjunctively merged possibility distribution of π_1 and π_2 is normal and there exists a $\omega \in \Omega$ such that $\omega \in B_{\pi_1} \cap B_{\pi_2}$. Recall that B_{π_1} and B_{π_2} are the respective smallest focal elements for m_1 and m_2, then for any A_{m_1} and A_{m_2}, two focal elements associated with m_1 and m_2 respectively, $A_{m_1} \cap A_{m_2} \neq \emptyset$. So $m_1 \oplus m_2(\emptyset) = 0$.

On the other hand, when $m_1 \oplus m_2(\emptyset) = 0$, $B_{\pi_1} \cap B_{\pi_2} \neq \emptyset$. Therefore, $\exists \omega$ such that $\omega \in B_{\pi_1} \cap B_{\pi_2}$. That is, $\pi_1(\omega) = \pi_2(\omega) = 1$ which implies $Inc(K) = 0$.

Now we prove $Inc(K) = 1$ iff $m_1 \oplus m_2(\emptyset) = 1$. When $Inc(K) = 1$, the conjunctively merged possibility distribution of π_1 and π_2 is totally inconsistent, then for any $\omega \in \Omega$ either $\pi_1(\omega) = 0$ or $\pi_2(\omega) = 0$ or both. Let $A^p_{m_1}$ and $A^q_{m_2}$ be the largest focal elements of m_1 and m_2 respectively, then $\omega \notin A^p_{m_1} \cap A^q_{m_2}$, so $A^p_{m_1} \cap A^q_{m_2} = \emptyset$. Therefore, for A_{m_1} and A_{m_2}, two focal elements associated with m_1 and m_2 respectively, $A_{m_1} \cap A_{m_2} = \emptyset$ which implies $m_1 \oplus m_2(\emptyset) = 1$.

Similar to this proof procedure, it is easy to show that when $m_1 \oplus m_2(\emptyset) = 1$, $Inc(B) = 1$.

Finally, we prove $Inc(K) > 0$ iff $m_1 \oplus m_2(\emptyset) > 0$. When $Inc(K) > 0$ there does not exist a $\omega \in \Omega$ such that $\omega \in B_{\pi_1} \cap B_{\pi_2}$ (otherwise $min(\pi_1(\omega), \pi_2(\omega)) = 1$ which violates the assumption). Since B_{π_1} and B_{π_2} are two smallest focal elements for m_1 and m_2 respectively, $B_{\pi_1} \cap B_{\pi_2} = \emptyset$ when combining these two mass functions, therefore $m(\emptyset) > 0$.

When $m(\emptyset) > 0$, we at least have $B_{\pi_1} \cap B_{\pi_2} = \emptyset$. So for any $\omega \in B_{\pi_1}$ (resp. B_{π_2}), it implies $\omega \notin B_{\pi_2}$ (resp. B_{π_1}), it follows immediately that $\min(\pi_1(\omega), \pi_2(\omega)) < 1$.
$\diamond$

In general conclusion $Inc(K_{12}) \geq Inc(K_{13}) \Rightarrow m_1 \oplus m_2(\emptyset) \geq m_1 \oplus m_3(\emptyset)$ does not hold.

4 Measuring conflict between PIBs

The conflict between uncertain information in possibility theory is measured by the degree of inconsistency induced by the information. However, this measure can only tell if two (or multiple) sources are inconsistent and to what extent, it cannot further differentiate pairs of PIBs that have the same degree of inconsistency.

Example 2 *Consider a set of four PIBs as detailed below with $\Omega = \{\omega_1, .., \omega_4\}$.*

$$K^1_1 = \{(\{\omega_1, \omega_2\}, 0.4), (\{\omega_2, \omega_3, \omega_4\}, 0.5), (\{\omega_2\}, 0.4)\}$$
$$K^1_2 = \{(\{\omega_1, \omega_2\}, 0.3), (\{\omega_1, \omega_2, \omega_3\}, 0.5), (\{\omega_1, \omega_4\}, 0.4)\}$$
$$K^1_3 = \{(\{\omega_1, \omega_3\}, 0.4), (\{\omega_2, \omega_3, \omega_4\}, 0.5), (\{\omega_3\}, 0.4)\}$$
$$K^1_4 = \{(\{\omega_2, \omega_4\}, 0.3), (\{\omega_1, \omega_3, \omega_4\}, 0.5), (\{\omega_1, \omega_4\}, 0.4)\}$$

Let $\pi^1_1, \pi^1_2, \pi^1_3$ and π^1_4 be the corresponding possibility distributions of these PIBs as detailed in Table 1.

Table 1. Four possibility distributions for the four PIBs.

PIB	π	ω_1	ω_2	ω_3	ω_4
K^1_1	π^1_1	0.5	1.0	0.6	0.6
K^1_2	π^1_2	1.0	0.6	0.6	0.5
K^1_3	π^1_3	0.5	0.6	1.0	0.6
K^1_4	π^1_4	0.6	0.5	0.6	1.0

Combining any pair of the four possibility distributions conjunctively (e.g., min) produces an unnormalized possibility distribution and in all the cases, the degree of *inconsistency is* 0.4 *(using min operator). It is, therefore, difficult to tell which two or more PIBs may be more consistent.*

In this section, we deploy two approaches developed in DS theory on measuring conflict among bbas to uncertain information in possibility theory.

4.1 A Distance-based measure of conflict

In [JGB01], a method for measuring the distance between bbas was proposed. This distance is defined as

$$d_{BPA}(m_1, m_2) = \sqrt{\frac{1}{2}(\tilde{m_1} - \tilde{m_2})^T \stackrel{D}{=} (\tilde{m_1} - \tilde{m_2})} \quad (8)$$

where $\stackrel{D}{=}$ is a $2^\Omega \times 2^\Omega$ dimensional matrix with $d[i, j] = |A \cap B|/|A \cup B|$ (note: it is defined that $|\emptyset \cap \emptyset|/|\emptyset \cup \emptyset| = 0$), and $A \in 2^\Omega$ and $B \in 2^\Omega$ are the names of columns and rows respectively. Given a bba m on frame Ω, $\tilde{m}$ is a 2^Ω-dimensional column vector (can also be called a $2^\Omega \times 1$ matrix) with $m_{A \in 2^\Omega}(A)$ as its 2^Ω coordinates.

$(\tilde{m_1} - \tilde{m_2})$ stands for vector subtraction and $(\tilde{m})^T$ is the transpose of vector (or matrix) $\tilde{m}$. When $\tilde{m}$ is a 2^Ω-dimensional column vector, $(\tilde{m})^T$ is its 2^Ω-dimensional row vector with the same coordinates. $((\tilde{m})^T \stackrel{D}{=} \tilde{m})$ therefore is the result of normal matrix multiplications (twice).

For example, let $\Omega = \{a, b\}$ be the frame and let $m(\{a\}) = 0.7, m(\Omega) = 0.3$ be a bba. Then $\tilde{m} = \begin{bmatrix} 0 \\ 0.7 \\ 0 \\ 0.3 \end{bmatrix}$ is a 4-dimensional column vector with row names ($\emptyset$, $\{a\}$, $\{b\}$, Ω) and $(\tilde{m})^T = [0, 0.7, 0, 0.3]$ is the corresponding row vector with column names ($\emptyset$, $\{a\}$, $\{b\}$, Ω). $\stackrel{D}{=}$ is a 4×4 square matrix with ($\emptyset$, $\{a\}$, $\{b\}$, Ω) as the names for both rows and columns. $((\tilde{m})^T \stackrel{D}{=} \tilde{m}) = 0.79$ in this example.

Example 3 (Continuing Example 2) *The four bbas recovered from the four possibility distributions in Example 2 are:*

$$\begin{array}{l} m_1(\{\omega_2\}) = 0.4,\ m_1(\{\omega_2, \omega_3, \omega_4\}) = 0.1,\ m_1(\Omega) = 0.5 \\ m_2(\{\omega_1\}) = 0.4,\ m_2(\{\omega_1, \omega_2, \omega_3\}) = 0.1,\ m_2(\Omega) = 0.5 \\ m_3(\{\omega_3\}) = 0.4,\ m_3(\{\omega_2, \omega_3, \omega_4\}) = 0.1,\ m_3(\Omega) = 0.5 \\ m_4(\{\omega_4\}) = 0.4,\ m_4(\{\omega_1, \omega_3, \omega_4\}) = 0.1,\ m_4(\Omega) = 0.5 \end{array}$$

Applying the distance-based measure defined in Equation 8 to all the pairs of PIBs, the distances between pairs of PIBs are listed below.

$$\begin{array}{lll} d_{BPA}(m_1, m_2) = 0.4203, & d_{BPA}(m_2, m_3) = 0.4203, & d_{BPA}(m_2, m_4) = 0.4203 \\ d_{BPA}(m_1, m_4) = 0.4358, & d_{BPA}(m_1, m_3) = 0.4, & d_{BPA}(m_3, m_4) = 0.4041 \end{array}$$

These results show that PIBs K_1 *and* K_4 *are most inconsistent whilst PIBs* (K_1, K_3) *or* (K_3, K_4) *are most consistent. This detailed analysis cannot be measured by the degree of inconsistency since every pair of PIBs has the same degree of inconsistency.*

A distance-based measure of a pair of bbas does not convey the same information as $m_1 \oplus m_2(\emptyset)$. More specifically, $Inc(K) = 0$ does not mean $d_{BPA} = 0$, nor does $Inc(K) = 1$ imply $d_{BPA} = 1$. For instance, a pair of possibility distributions π_1 and π_2 defined on $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ for two PIBs with

$$\pi_1(\omega_1) = 1,\ \pi_1(\omega_2) = 0.5,\ \pi_1(\omega_3) = 0.4,\ \pi_1(\omega_4) = 0.4$$
$$\pi_2(\omega_1) = 1,\ \pi_2(\omega_2) = 1,\quad \pi_2(\omega_3) = 1,\quad \pi_2(\omega_4) = 0.8$$

produces a normal possibility distribution after a conjunctive merge. The degree of inconsistency is $Inc(K_{12}) = 0$ where K_{12} is the merged PIB. However, $d_{BPA}(m_1, m_2) = 0.41$ where m_1 and m_2 are the bbas for π_1 and π_2. Similarly, if we have a pair of possibility distributions π_3 and π_4 defined on the same set Ω as

$$\pi_3(\omega_1) = 1,\ \pi_3(\omega_2) = 0.6,\ \pi_3(\omega_3) = 0,\ \pi_3(\omega_4) = 0$$
$$\pi_4(\omega_1) = 0,\ \pi_4(\omega_2) = 0,\quad \pi_4(\omega_3) = 1,\ \pi_4(\omega_4) = 0.8$$

then $Inc(K_{34}) = 1$ whilst $d_{BPA}(m_3, m_4) = 0.842$ where K_{34} is the merged PIB and m_3 and m_4 are the bbas for π_3 and π_4 respectively.

This discussion shows that the distance-based measure can not replace the measure of degree of inconsistency. Both measures should be used when assessing how conflict a pair of PIBs is.

4.2 A (difBetP, $m_1 \oplus m_2(\emptyset)$) based measure of conflict

The conflict between two belief functions (or bbas) in DS theory is traditionally measured using the combined mass value assigned to the emptyset before normalization, e.g., $m(\emptyset)$. In [Liu06], it is illustrated that this measure is not accurate and a new measure which is made up of two values is introduced. One of these two values is the difference between betting commitments obtained through pignistic probability functions and another is the combined value assigned to the emptyset before normalization.

Definition 3. *(adapted from [Liu06]) Let m_1 and m_2 be two bbas on Ω and $BetP_{m_1}$ and $BetP_{m_2}$ be their corresponding pignistic probability functions. Then*

$$\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} = \mathsf{max}_{\omega \in \Omega}(|BetP_{m_1}(\omega) - BetP_{m_2}(\omega)|)$$

is called the distance between betting commitments *of the two bbas.*

Value $(|BetP_{m_1}(\omega) - BetP_{m_2}(\omega)|)$ is *the difference between betting commitments to possible world* ω from the two sources. The distance of betting commitments, $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}}$, is therefore the maximum extent of the differences between betting commitments to all the possible worlds. This definition is a revised version in [Liu06] where in the original definition for $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}}$, ω is replaced by A (a subset). The rational for this adaptation is that we want to know how "far apart" the degrees of possibility assigned to a possible world is from the two sources.

We use the following example to show the advantage of (difBetP, $m_1 \oplus m_2(\emptyset)$) over $m_1 \oplus m_2(\emptyset)$.

Example 4 *Let m_1 and m_2 be two bbas on $\Omega = \{\omega_1, ..., \omega_5\}$ as*

$$m_1(\{\omega_1\}) = 0.8,\ m_1(\{\omega_2, \omega_3, \omega_4, \omega_5\}) = 0.2,$$

and

$$m_2(\Omega) = 1.$$

Then $m_1 \oplus m_2(\emptyset) = 0$ when m_1 and m_2 are combined with Dempster's rule, which is traditionally explained as there is no conflict between the two bbas. However, m_1 is more committed whilst m_2 is less sure about its belief as which value(s) are more plausible than others. The difference in their opinions is reflected by $\mathsf{difBetP}_{m_1}^{m_2} = 0.6$. It says that the two sources have rather different beliefs as where the true hypothesis lies.

Definition 4. *Let (K_1, K_2) and (K_1, K_3) be two pairs of PIBs and K_{12} and K_{13} be the two merged PIBs from these two pairs. Let m_1, m_2, and m_3 be the bbas for the three PIBs respectively. Assume that $Inc(K_{12}) = Inc(K_{13})$, then K_1 is more consistent with K_2 than with K_3 when the following condition holds*

$$\mathsf{difBetP}_{\mathsf{m}_1}^{\mathsf{m}_2} \leq \mathsf{difBetP}_{\mathsf{m}_1}^{\mathsf{m}_3} \text{ and } m_1 \oplus m_2(\emptyset) \leq m_1 \oplus m_3(\emptyset)$$

Example 5 *Let three PIBs on set $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ be*

$$K_1^2 = \{(\{\omega_1, \omega_3\}, 0.4), (\{\omega_2, \omega_3, \omega_4\}, 0.5), (\{\omega_2\}, 0.4)\}$$
$$K_2^2 = \{(\{\omega_1, \omega_2\}, 0.3), (\{\omega_1, \omega_2, \omega_3\}, 0.5), (\{\omega_1, \omega_4\}, 0.4)\}$$
$$K_3^2 = \{(\{\omega_1, \omega_2, \omega_3\}, 0.4), (\{\omega_1, \omega_2, \omega_4\}, 0.4), (\{\omega_2, \omega_3\}, 0.4)\}$$

The corresponding possibility distributions and bbas for these PIBs are

$$\pi_1^2(\omega_1) = 0.5,\ \pi_1^2(\omega_2) = 0.6,\ \pi_1^2(\omega_3) = 1.0,\ \pi_1^2(\omega_4) = 0.6,$$
$$\pi_2^2(\omega_1) = 1.0,\ \pi_2^2(\omega_2) = 0.6,\ \pi_2^2(\omega_3) = 0.6,\ \pi_2^2(\omega_4) = 0.5,$$
$$\pi_3^2(\omega_1) = 0.6,\ \pi_3^2(\omega_2) = 1.0,\ \pi_3^2(\omega_3) = 0.6,\ \pi_3^2(\omega_4) = 0.6.$$

and

$$m_1^2(\{\omega_3\}) = 0.4,\ m_1^2(\{\omega_2, \omega_3, \omega_4\}) = 0.1,\ m_1^2(\Omega) = 0.5$$
$$m_2^2(\{\omega_1\}) = 0.4,\ m_2^2(\{\omega_1, \omega_2, \omega_3\}) = 0.1,\ m_1^2(\Omega) = 0.5$$
$$m_3^2(\{\omega_2\}) = 0.4,\ m_3^2(\Omega) = 0.6,$$

$Inc(K_{12}^2) = Inc(K_{13}^2) = 0.4$. *However,* $m_1^2 \oplus m_2^2(\emptyset) = 0.20$ *and* $m_1^2 \oplus m_3^2(\emptyset) = 0.16$. *Furthermore,*

$$\mathsf{difBetP}_{m_1^2}^{m_2^2} = 0.4 + 0.1/3, \text{ and } \mathsf{difBetP}_{m_1^2}^{m_3^2} = 0.4 + 0.1/4 - 0.1/3$$

Therefore,

$$\mathsf{difBetP}_{m_1^2}^{m_3^2} < \mathsf{difBetP}_{m_1^2}^{m_2^2}$$

and

$$m_1^2 \oplus m_3^2(\emptyset) < m_1^2 \oplus m_2^2(\emptyset)$$

K_1^2 *is more consistent with* K_3^2 *than with* K_2^2.

In [Liu06], it has been shown that the ($\mathsf{difBetP}$, $m_1 \oplus m_2(\emptyset)$) based approach is more appropriate to measure the conflict among evidence than the distance-based approach. This can at least be seen from re-examining Example 2 using ($\mathsf{difBetP}$, $m_1 \oplus m_2(\emptyset)$). For example, applying this approach to the first pair of bbas derived from (π_1, π_2) in Example 2, we have ($\mathsf{difBetP}_{m_1}^{m_2}$, $m_1 \oplus m_2(\emptyset)$)=(0.383, 0) which concludes that the two pieces of information are largely consistent (since $m_1 \oplus m_2(\emptyset) = 0$) but there is some disagreement among them (since $\mathsf{difBetP}_{m_1}^{m_2} \neq 0$). However, the degree of inconsistency (which is 0) as a single value cannot give us this (further) information.

5 Assessment of agent's judgement

When pieces of uncertain information are highly inconsistent and they have to be merged, some resolutions are needed before a meaningful merged result can be obtained. One common approach is to make use of the reliability of a source, so that the information from a source with a lower reliability can be either discarded or discounted (e.g., weakened). However, reliabilities are often required as extra knowledge and this knowledge is not always readily available. Therefore, finding ways of assessing the reliability of a source is the first step towards how to handle highly conflicting information.

In [DP94], a method for assessing the quality of information provided by a source was proposed. This method is to measure how accurate and informative the provided information is.

Let x be a (testing) variable for which all the possible values are included in set Ω and its true value (denoted as v) is known. To assess the reliability of a source (hereafter referred to as *Agent*), *Agent* is asked to provide its judgement as what is the true value for x. Assume that *Agent*'s reply is a set of weighted nested subsets in terms of possibility theory

$$K = \{(A_1, \alpha_1), ..., (A_n, \alpha_n)\} \texttt{ where } A_i \subset A_j, i < j$$

Then a possibility distribution π_x as well as a bba m can be constructed from this information on Ω such that

$$\begin{array}{ll} \pi_x(\omega) = \beta_1 = 1 & \texttt{when } \omega \in A_1 \\ \pi_x(\omega) = \beta_2 & \texttt{when } \omega \in A_2 \setminus A_1 \texttt{ and } \beta_2 = 1 - \alpha_1 \\ \pi_x(\omega) = \beta_3 & \texttt{when } \omega \in A_3 \setminus A_2 \texttt{ and } \beta_3 = 1 - \alpha_2 \\ \vdots & \\ \pi_x(\omega) = \beta_n & \texttt{when } \omega \in A_n \setminus A_{n-1} \texttt{ and } \beta_n = 1 - \alpha_{n-1} \\ \pi_x(\omega) = \beta_{n+1} & \texttt{when } \omega \notin A_n; \texttt{ and } \beta_{n+1} = 1 - \alpha_n \end{array}$$

Then $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_{n+1}$, since $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ due to the monotonicity of N and

$$m(A_1) = \beta_1 - \beta_2,\ m(A_2) = \beta_2 - \beta_3,\ \ldots,\ m(A_n) = \beta_n - \beta_{n+1}$$

The rating of *Agent*'s judgement in relation to this variable is therefore defined as [DP94]

$$Q(K, x) = \pi_x(v) \frac{|\Omega| - ||K||}{(1 - m(A_n))|\Omega|} \tag{9}$$

where $||K|| = \Sigma_{i=1}^{m}(|A_i|m(A_i))$, v is the actual value of variable x, and $|\Omega|$ (resp. $|A_i|$) is the cardinality of set Ω (resp. A_i). This formula ensures that *Agent* can score high only if he is both accurate (with a high $\pi_x(v)$) and informative (with a fairly focused subset).

When $K = \{(\Omega, 1)\}$, it implies $m(\Omega) = 1$ and $\pi_x(\omega) = 1, \forall\omega \in \Omega$, then $Q(K, x) = 0$ since $||K|| = |\Omega|$. This shows that the *Agent* is totally ignorant. When $K = \{(\{v\}, 1)\}$, it implies $\pi_x(v) = 1$ and $\pi_x(\omega) = 0$ when $\omega \neq v$. Then $Q(K, x) = (|\Omega| - 1)/|\Omega|$ since $m(A_n) = 0$. This conclusion says that the *Agent*'s judgement increases along the size of the set of all values, the bigger the set, the more accurate the *Agent*'s judgement is.

When the *Agent*'s reply is not in the form of a set of weighted nested subsets, relationships between DS theory and possibility theory studies in Section 3 should be used to construct a set of nested subsets, called focal elements. Then this set of nested subsets can be used in Equation 9 for calculating the ranking of an *Agent*.

The overall rating of an *Agent* is evaluated as the average of all ratings obtained from answering a set of (testing) variables where *Agent*'s reply for each variable is judged using Equation 9. Once each *Agent*'s rating is established, suitable discounting operators ([DP01]) can be applied to weaken the opinions from less reliable *Agents* to resolve inconsistency among information.

6 Conclusion

In this paper, we have shown that additional approaches to measuring inconsistency among pieces of uncertain information are needed since the only measure used in possibility theory, e.g., the degree of inconsistency, is not adequate for situations where pairs of uncertain information have the same degree of inconsistency. We have preliminarily investigated how two recently proposed methods in DS theory on inconsistency/conflict measures can be used to measure the inconsistency among pieces of uncertain information in possibility theory. In addition, we have also looked at issues as how the reliability (or judgement) of a source can be established through assessing the quality of answers to a set of known situations.

All these studies will have an impact on which merging operator should be selected for what conflict scenario and how inconsistencies should be resolved if reliabilities of sources are known. We will investigate all these issues in depth in a future paper.

In [HL05a,HL05b], a coherence interval based method was proposed to quantitatively measure how consistent a pair of uncertain possibilistic information is. This method clearly offers a very different alternative to the two methods developed in DS theory. Comparing these three alternatives will be another objective for our future research.

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