Quota-based merging operators for stratified knowledge bases

Guilin Qi, Weiru Liu, David A. Bell

School of Electronics, Electrical Engineering and Computer Science
Queen's University Belfast, Belfast, BT7 1NN, UK
{G.Qi,W.Liu,DA.Bell}@qub.ac.uk

Abstract. Current merging methods for stratified knowledge bases are often based on the commensurability assumption, i.e. all knowledge bases share a common scale. However, this assumption is too strong in practice. In this paper, we propose a family of operators to merge stratified knowledge bases without commensurability assumption. Our merging operators generalize the quota operators, a family of important merging operators in classical logic. Both logical properties and computational complexity issues of the proposed operators are studied.

1 Introduction

The problem of merging multiple sources of information is important in many applications, such as database merging [14] and group decision making [15]. Priorities, either implicit or explicit, play an important role in belief merging. In classical logic, a knowledge base is a set of formulas with the same level of priority. However, an implicit ordering on the set of possible worlds can be extracted from it [11, 14]. In some cases, we even assume that explicit priorities are attached to each source which takes the form of a stratified set of beliefs or goals [8, 20]. That is, each source can be viewed as a stratified or prioritized knowledge base.

Merging of stratified knowledge bases is often handled in the framework of possibilistic logic [8] or ordinal conditional function [20]. Usually, the merging methods are based on the assumption that all agents use the same scale (usually ordinal scales such as [0,1]) to order their beliefs. However, in practice, the numerical information is hard to get-we may only have a knowledge base with a total pre-order relation on its formulas. In addition, different agents may use different ways to order their beliefs. Even a single agent may have different ways of modeling her preferences for different aspects of a problem [6]. In that case, the previous merging methods cannot be applied.

It is widely accepted that belief merging is closely related to social choice theory [15, 7, 13, 9]. In social choice theory, we have a group of p voters (or agents). Each voter suggests a preference on a set of alternatives. An important problem is then to define a voting rule which is a function mapping a set of p preferences to an alternative or a set of alternatives. Many voting rules have been proposed, such as the Plurality rule [16] and the voting by quota [2].

In this paper, we propose a family of quota-based merging operators for stratified knowledge bases under integrity constraints. We assume that each stratified knowledge base is assigned to an ordering strategy. For each stratified knowledge base K and its ordering strategy X, we get a complete, transitive and asymmetric preference relation $<_{K,X}$ on subsets of the set of possible worlds. A possible world is a model of the resulting knowledge base of the quota-based merging operator if it belongs to the most preferred element of at least k preference relations. The quota-based merging operators are problematic in some cases. So we define a refined version of the quota-based merging operators.

This paper is organized as follows. Some preliminaries are given in Section 2. Section 3 introduces quota merging operators in propositional logic. In Section 4, we consider the preference representation of stratified knowledge bases. A new ordering strategy is proposed. Our merging operators are defined in Section 5. Section 6 analyzes the computational complexity of our merging operators. We then study the logical properties of our merging operators in Section 7. Section 8 discusses related work. Finally, we conclude the paper in Section 9.

2 Preliminaries

Classical logic: In this paper, we consider a propositional language $\mathcal{L}_{PS}$ from a finite set PS of propositional symbols. The classical consequence relation is denoted as $\vdash$. An interpretation (or possible world) is a total function from PS to $\{0, 1\}$, denoted by a bit vector whenever a strict total order on PS is specified. Ω is the set of all possible interpretations. An interpretation w is a model of a formula ϕ iff $w(\phi) = 1$. p, q, r,... represent atoms in PS. We denote formulas in $\mathcal{L}_{PS}$ by ϕ, ψ, γ,... For each formula ϕ, we use $M(\phi)$ to denote its set of models. A *classical knowledge base* K is a finite set of propositional formulas (we can also identify K with the conjunction of its elements). K is consistent iff there exists an interpretation w such that $w(\phi) = true$ for all $\phi \in K$. A knowledge *profile* E is a multi-set of knowledge bases, i.e. $E = \{K_1, ..., K_n\}$, where K_i may be identical to K_j for $i \neq j$. Let $\bigcup(E) = \cup_{i=1}^{n} K_i$. Two knowledge profiles E_1 and E_2 are equivalent, denoted $E_1 \equiv E_2$ iff there exists a bijection f between E_1 and E_2 such that for each $K \in E_1$, $f(K) \equiv K$.

Stratified knowledge base: A *stratified* knowledge base, sometimes also called ranked knowledge base [6] or prioritized knowledge base [3], is a set K of (finite) propositional formulas together with a total preorder $\leq$ on K (a preorder is a transitive and reflexive relation, and $\leq$ is a total preorder if either $\phi \leq \psi$ or $\psi \leq \phi$ holds for any $\phi, \psi \in K$)[1]. Intuitively, if $\phi \leq \psi$, then ϕ is considered to be less important than ψ. K can be equivalently defined as a sequence $K = (S_1, ..., S_n)$, where each S_i $(i = 1, ..., n)$ is a non-empty set which contains all the maximal elements of $K \setminus (\cup_{j=1}^{i-1} S_j)$ w.r.t $\leq$, i.e. $S_i = \{\phi \in K \setminus (\cup_{j=1}^{i-1} S_j) : \forall \psi \in K \setminus (\cup_{j=1}^{i-1} S_j), \psi \leq \phi\}$. Each subset S_i is called a stratum of K and i the

[1] For simplicity, we use K to denote a stratified knowledge base and ignore the total preorder $\leq$.

priority level of each formula of S_i. Therefore, the lower the stratum, the higher the priority level of a formula in it. A stratified knowledge profile (SKP) E is a multi-set of stratified knowledge bases. Given a stratified knowledge base $K = (S_1, ..., S_n)$, the i-cut of K is defined as $K_{\geq i} = S_1 \cup ... \cup S_i$, for $i \in \{1, ..., n\}$. A subbase A of K is also stratified, that is, $A = (A_1, ..., A_n)$ such that $A_i \subseteq S_i$, $i = 1, ..., n$. Two SKPs E_1 and E_2 are equivalent, denoted $E_1 \equiv_s E_2$ iff there exists a bijection between E_1 and E_2 such that $n = m$ and for each $K = (S_1, ..., S_l) \in E_1$, $f(K) = (S'_1, ..., S'_l)$ and $S_i \equiv S'_i$ for all $i \in \{1, ..., l\}$.

3 Quota Merging Operator

In this section, we introduce the quota operators defined in [9].

Definition 1. *[9] Let k be an integer, $E = \{K_1, ..., K_n\}$ be a multi-set of knowledge bases, and μ be a formula. The k-quota merging operator, denoted Δ^k, is defined in a model-theoretic way as:*

$$M(\Delta^k_\mu(E)) = \begin{cases} \{\omega \in M(\mu) | \forall K_i \in E \ \omega \models K_i\} & \text{if not empty,} \\ \{\omega \in M(\mu) | \sharp(\{K_i \in E | \ \omega \models K_i\}) \geq k\} & \text{otherwise.} \end{cases} \quad (1)$$

(#L denotes the number of the elements in L.)

The resulting knowledge base of the k-quota merging of E under constraints μ is simply the conjunction of the bases when $\bigwedge E \wedge \mu$ is consistent. Otherwise, the models of the resulting knowledge base are the models of μ which satisfy at least k bases of E.

The choice of an appropriate k is very important to define a good quota merging operator. An interesting value of k is the maximum value such that the merged base is consistent. That is, we have the following definition.

Definition 2. *[9] Let $E = \{K_1, ..., K_n\}$ be a knowledge profile, and μ be a formula. Let $k_{max} = max(\{i \leq \sharp(E) | \Delta^i_\mu \not\models \bot\})$. $\Delta^{k_{max}}$ is defined in a model-theoretical way as:*

$$M(\Delta^{k_{max}}_\mu(E)) = \begin{cases} \{\omega \in M(\mu) | \forall K_i \in E \ \omega \models K_i\} & \text{if not empty,} \\ \{\omega \in M(\mu) | \sharp(\{K_i \in E | \ \omega \models K_i\}) = k_{\max}\} & \text{otherwise.} \end{cases} \quad (2)$$

4 Preference Representation of Stratified Knowledge Bases

4.1 Ordering strategies

Given a stratified knowledge base $K = \{S_1, ..., S_n\}$, we can define some total pre-orders on Ω.

- **best out ordering** [3]:
  Let $r_{BO}(\omega) = min\{i : \omega \not\models S_i\}$, for $\omega \in \Omega$. Then the best out ordering $\preceq_{bo}$ on Ω is defined as: $\omega \preceq_{bo} \omega' \ iff \ r_{BO}(\omega) \geq r_{BO}(\omega')$

- **maxsat ordering** [6]:
  Let $r_{MO}(\omega) = min\{i : \omega \models S_i\}$, for $\omega \in \Omega$. Then the maxsat ordering $\preceq_{maxsat}$ on Ω is defined as: $\omega \preceq_{maxsat} \omega' \ iff \ r_{MO}(\omega) \leq r_{MO}(\omega')$
- **leximin ordering** [3]:
  Let $K^i(\omega) = \{\phi \in S_i : \omega \models \phi\}$. Then the leximin ordering $\preceq_{leximin}$ on Ω is defined as:
  $\omega \preceq_{leximin} \omega'$ iff $|K^i(\omega)| = |K^i(\omega')|$ for all i, or there is an i such that $|K^i(\omega')| < |K^i(\omega)|$, and for all $j < i$: $|K^j(\omega)| = |K^j(\omega')|$, where $|K_i|$ denote the cardinality of the sets K_i.

Given a preorder $\preceq$ on Ω, as usual, the associated strict partial order is defined by $\omega \prec \omega'$ iff $\omega \preceq \omega'$ and not $\omega' \preceq \omega$. An ordering $\preceq_X$ is more *specific* than another $\preceq_{X'}$ iff $\omega \prec_{X'} \omega'$ implies $\omega \prec_X \omega'$. The total preorders on Ω defined above are not independent of each other.

Proposition 1. *[6] Let $\omega, \omega' \in \Omega$, K a stratified knowledge base. The following relationships hold: $\omega \prec_{bo} \omega'$ implies $\omega \prec_{leximin} \omega'$;*

4.2 A new ordering strategy

We now define a new ordering strategy by considering the "distance" between an interpretation and a knowledge base.

Definition 3. *[9] A pseudo-distance between interpretations is a total function d from $\Omega \times \Omega$ to N such that for every ω_1, $\omega_2 \in \Omega$: (1) $d(\omega_1, \omega_2) = d(\omega_2, \omega_1)$; and (2) $d(\omega_1, \omega_2) = 0$ if and only if $\omega_1 = \omega_2$.*

A "distance " between an interpretation ω and a knowledge base S can then be defined as $d(\omega, S) = min_{\omega' \models S} d(\omega, \omega')$. When S is inconsistent, $d(\omega, S) = +\infty$. That is, all the possible worlds have the same distance with an inconsistent knowledge base. Two common examples of such distances are the *drastic distance* d_D and the *Dalal distance* d_H, where $d_D(\omega_1, \omega_2) = 0$ when $\omega_1 = \omega_2$ and 1 otherwise, and $d_H(\omega_1, \omega_2)$ is the Hamming distance between ω_1 and ω_2.

Definition 4. *The distance-based ordering $\preceq_d$ on Ω is defined as:*
$\omega \preceq_d \omega'$ iff $d(\omega, S_i) = d(\omega', S_i)$ for all i, or there is an i such that $d(\omega, S_i) < d(\omega', S_i)$, and for all $j < i$: $d(\omega, S_j) = d(\omega', S_j)$.

It is clear that the distance-based orderings are total preorders on Ω. Suppose $d = d_H$, the ordering $\preceq_{d_H}$ is equivalent to the total preorder $\leq_{K,Lex}$ which is defined to characterize the minimal change of a revision operator in [17].

Proposition 2. *Let $\omega, \omega' \in \Omega$, and K be a stratified knowledge base. Suppose $d = d_D$ or d_H, then we have: (1) $\omega \preceq_d \omega'$ implies $\omega \preceq_{bo} \omega'$ and $\omega \preceq_d \omega'$; (2) $\omega \prec_{bo} \omega'$ implies $\omega \prec_d \omega'$*

5 Quota-based Merging Operators

5.1 Voting by quota

Let A be a finite set of objects and $N = \{1, 2, ..., n\}$ be a set of n voters (or agents), where $n \geq 2$. *Alternatives* are subsets of A. We use X, Y and Z to denote alternatives. The ith voter's preference relations, denoted by $\prec_i$, $\prec_i^{'}$, etc, are complete, transitive, and asymmetric relations on 2^A (the set of subsets of A). For $X, Y \in 2^A$, $X \prec_i Y$ means X is strictly preferred to Y $w.r.t$ voter i. Let $\mathcal{X} \subseteq 2^A$, we denote by $min(\mathcal{X}, \prec_i)$ the most preferred alternative in $\mathcal{X}$ according to $\prec_i$. Let P denote the set of all preference relations on A. A voting rule on the domain $D_1 \times ... \times D_n \subseteq P^n$ is a function $f : D_1 \times ... \times D_n \rightarrow A$, where each D_i is considered to represent the set of ith voter's preference relations.

We now introduce a voting rule, called voting by quota.

Definition 5. *[2] A vote rule* $f : D_1 \times ... \times D_n \rightarrow 2^A$ *is* voting by quota *if there exists* k *between 1 and* n *such that for all* $(\prec_1, ..., \prec_n)$*, we have* $x \in f(\prec_1, ..., \prec_n)$ *if and only if* $\#\{i | x \in min(2^A, \prec_i)\} \geq k$.

Voting by quota k selects the alternative consisting of objects which are in at least k most preferred alternatives of 2^A according to $\prec_i$.

5.2 Quota-based merging operator

We use $\preceq_X$ to denote a total preorder on Ω, where X represents an ordering strategy. For example, if $X = bo$, then $\preceq_X$ is the best-out ordering. The idea of defining our quota-based operators can be explained as follows. First, for each stratified knowledge base K_i and the ordering strategy X_i, we obtain a complete, transitive and asymmetric preference relation on 2^Ω. We then apply voting by quota to aggregate the preferences and the obtained set of possible worlds is taken as the set of models of the resulting knowledge base.

Given a stratified knowledge base K and an ordering strategy X, Ω can be stratified with regard to the total preorder $\preceq_X$ on it as $\Omega_{K,X} = (\Omega_1, ..., \Omega_m)$ in the same way as stratifying a knowledge base. For two interpretations ω_1, ω_2, if $\omega_1 \in \Omega_i$ and $\omega_2 \in \Omega_j$, where $i < j$, then ω_1 is preferred to ω_2. A complete, transitive and asymmetric preference relation $<_{K,X}$ on 2^Ω can then be defined as follows. (1) For $W, W' \in 2^\Omega$, if $W = \Omega_i$ and $W' = \Omega_j$, where $i < j$, then $W <_{K,X} W'$; if $W = \Omega_i$ for some i, and there does not exist j such that $W' = \Omega_j$, then $W <_{K,X} W'$; (2) For elements in $2^\Omega \setminus \{\Omega_1, ..., \Omega_n\}$, we order them as $W \leq_{K,X} W'$ iff $\forall i$, $\#(W \cap \Omega_i) = \#(W' \cap \Omega_i)$ or $\exists i$ such that $\#(W \cap \Omega_i) > \#(W' \cap \Omega_i)$ and $\#(W \cap \Omega_j) = \#(W' \cap \Omega_j)$ for all $j < i$. It is possible that there exist some W_i $(i = 1, ..., k)$ such that $W_i =_{K,X} W_j$ for any pair i and j, where $W_i =_{K,X} W_j$ means $W \leq_{K,X} W'$ and $W' \leq_{K,X} W$. In that case, we arbitrary order them as $W_1, W_2, ..., W_k$ such that $W_i <_{K,X} W_j$ if $i < j$. (3) Finally, for all $W, W' \in 2^\Omega$, if $W <_{K,X} W'$, then not $W' <_{K,X} W$. It is easy to check that $<_{K,X}$ defined above is a complete, transitive and asymmetric relation on 2^Ω.

Definition 6. *Let $E = \{K_1, ..., K_n\}$ be a multi-set of stratified knowledge bases, where $K_i = \{S_{i1}, ..., S_{im}\}$, μ be a formula, and let k be an integer. Let $\mathbf{X} = (X_1, ..., X_n)$ be a set of ordering strategies, where X_i ($i = 1, ..., n$) are ordering strategies attached to K_i. Suppose $<_{K_i,X_i}$ is the complete, transitive and asymmetric relation on 2^Ω obtained by K_i and X_i. The resulting knowledge base of k-quota merging operator, denoted by $\Delta_\mu^{k,\mathbf{X}}(E)$, is defined in a model-theoretic way as follows:*

$$M(\Delta_\mu^{k,\mathbf{X}}(E)) = \{\omega \in M(\mu) | \sharp(\{K_i \in E | \ \omega \in Min(2^\Omega, <_{K_i,X_i})\}) \geq k\}.$$

The models of the resulting knowledge base of the k-quota merging of E under constraints μ are the models of μ which most preferred according to at least k preference relations.

Example 1. Let $E = \{K_1, K_2, K_3\}$ be a SKP consisting of three stratified knowledge bases, where

- $K_1 = \{S_{11}, S_{12}, S_{13}\}$, where $S_{11} = \{p_1 \vee p_2, p_3\}$, $S_{12} = \{\neg p_1, \neg p_2, p_2 \vee \neg p_3, p_4\}$, $S_{13} = \{\neg p_3 \vee \neg p_4\}$
- $K_2 = \{S_{21}, S_{22}\}$, where $S_{21} = \{p_1, p_2 \vee p_3\}$ and $S_{22} = \{\neg p_2, p_4\}$
- $K_3 = \{S_{31}, S_{32}\}$, where $S_{31} = \{p_1, p_3\}$ and $S_{32} = \{p_2\}$.

The integrity constraint is $\mu = \{\neg p_1 \vee p_2\}$. The set of models of μ is $M(\mu) = \{\omega_1 = 0111, \omega_2 = 0101, \omega_3 = 0110, \omega_4 = 0100, \omega_5 = 0011, \omega_6 = 0001, \omega_7 = 0010, \omega_8 = 0000, \omega_9 = 1111, \omega_{10} = 1101, \omega_{11} = 1110, \omega_{12} = 1100\}$. We denote each model by a bit vector consisting of truth values of (p_1, p_2, p_3, p_4). For example, $\omega_1 = 0111$ means that the truth value of p_1 is 0 and the truth values of other atoms are all 1. Let $\mathbf{X} = \{X_1, X_2, X_3\}$, where $X_1 = X_2 = bo$ and $X_3 = d_H$. That is, the best out ordering strategy is chosen for both K_1 and K_2, whilst the Dalal distance-based ordering is chosen for K_3. The computations are given in Table 1 below.

ω	K_1	K_2	K_3
0111	1	3	3
0101	2	3	5
0110	1	3	3
0100	2	3	5
0011	2	3	4
0001	2	3	6
0010	2	3	4
0000	2	3	6
1111	1	2	1
1101	2	2	3
1110	1	2	1
1100	2	2	3

Table 1

In Table 1, the column corresponding to K_i gives the priority levels of strata of Ω_{K_i,X_i} where ω_i belongs to. Let us explain how to obtain the column corresponding to K_2 (other columns can be obtained similarly). Let $\omega_{13} = 1011$, $\omega_{14} = 1001$, $\omega_{15} = 1010$ and $\omega_{16} = 1000$. Since $r_{BO}(\omega_i) = 1$ for all $1 \leq i \leq 8$, $r_{BO}(\omega_i) = 2$ for $9 \leq i \leq 12$ and $14 \leq i \leq 16$, $r_{BO}(\omega_{13}) = +\infty$, we have $\Omega_{K_2,bo} = (\{\omega_{13}\}, \{\omega_9, ..., \omega_{12}, \omega_{14}, ..., \omega_{16}\}, \{\omega_1, ..., \omega_8\})$. So $l_{K_2,bo}(\omega_i) = 3$ for $1 \leq i \leq 8$ and $l_{K_2,bo}(\omega_i) = 2$ for $9 \leq i \leq 12$. Let k=1. Since ω_1, ω_3, ω_9 and ω_{11} are the only models of μ which belong to the level 1 of the strata of at least one of Ω_{K_i,X_i}, we have $M(\Delta^1_\mu(E)) = \{0111, 0110, 1111, 1110\}$. Let $k = 3$. Since none of models of μ is in the first level of strata of all Ω_{K_i,X_i} $(i = 1, 2, 3)$, we have $M(\Delta^3_\mu(E)) = \emptyset$.

By Example 1, the resulting knowledge base of the k-quota based merging operator may be inconsistent.

Clearly, we have the following proposition.

Proposition 3. *Let k be an integer, $E = \{K_1, ..., K_n\}$ be a multi-set of knowledge bases, and μ be a formula. Let $\mathbf{X} = (X_1, ..., X_n)$ be a set of ordering strategies, where X_i $(i = 1, ..., n)$ are ordering strategies attached to K_i. We have $\Delta^{k+1,\mathbf{X}}_\mu(E) \models \Delta^{k,\mathbf{X}}_\mu(E)$ or equivalently, $M(\Delta^{k+1,\mathbf{X}}_\mu(E)) \subseteq M(\Delta^{k,\mathbf{X}}_\mu(E))$. The converse does not generally hold.*

According to Proposition 3, the quota-based operators lead to a sequence of merged bases that is monotonic *w.r.t.* logical entailment. That is, the number of models of the merged bases may decrease when k increases. So the set of models of the merged bases may be empty for some k. We have the following definition which generalizes the k_{max}-quota operator.

Definition 7. *Let $E = \{K_1, ..., K_n\}$ be a SKP, and μ be a formula. Let $k_{max} = max(\{i \leq \sharp(E) | \Delta^{i,\mathbf{X}}_\mu \not\models \perp\})$. $\Delta^{k_{max},\mathbf{X}}$ is defined in a model-theoretical way as:*

$$M(\Delta^{k_{max},\mathbf{X}}_\mu(E)) = \{\omega \in M(\mu) | \sharp(\{K_i \in E | \ \omega \in Min(2^\Omega, <_{K_i,X_i})\}) = k_{\max}\}.$$

Example 2. (continue Example 1) $k_{max} = 2$. So the result of merging by the $\Delta^{k_{max},\mathbf{X}}$ operator is $M(\Delta^{k_{max},\mathbf{X}}_\mu(E)) = \{1111, 1110\}$. That is, $\Delta^{k_{max},\mathbf{X}}_\mu(E) = p_1 \wedge p_2 \wedge p_3$.

The following proposition states the relationship between different $\Delta^{k,\mathbf{X}}$ operators when considering different ordering strategies.

Proposition 4. *Let $E = \{K_1, ..., K_n\}$ be a SKP, μ be the integrity constraint, and let k be an integer. Let $\mathbf{X_1} = \{X_1, ..., X_n\}$ and $\mathbf{X_2} = \{X'_1, ..., X'_n\}$ be two vectors of ordering strategies, where both X_i and X'_i are ordering strategies for K_i. Suppose $\preceq_{X_i}$ is more specific than $\preceq_{X'_i}$, for all i, where $X_i \in \mathbf{X_1}$ and $X'_i \in \mathbf{X_2}$, then $\Delta^{k,\mathbf{X_2}}_\mu(E) \models \Delta^{k,\mathbf{X_1}}_\mu(E)$.*

Proposition 4 shows that the operator with regard to the set of more specific ordering strategies can result in a knowledge base which has stronger inferential power.

5.3 Refined quota-based merging operator

The quota-based operators is problematic when merging knowledge bases which are jointly consistent with the formula representing the integrity constraints, i.e. $K_1 \cup ... \cup K_n \cup \phi$ is consistent.

Example 3. Let $E = \{K_1, K_2, K_3\}$ be a SKP consisting of three stratified knowledge bases, where

- $K_1 = \{S_{11}, S_{12}\}$, where $S_{11} = \{p_1 \vee p_2, p_3\}$, $S_{12} = \{\neg p_1, p_4\}$
- $K_2 = \{S_{21}, S_{22}\}$, where $S_{21} = \{p_2 \vee p_3\}$ and $S_{22} = \{p_4\}$
- $K_3 = \{S_{31}, S_{32}\}$, where $S_{31} = \{p_3\}$ and $S_{32} = \{p_2\}$.

The integrity constraint is $\mu = \{\neg p_1 \vee p_2\}$. The set of models of μ is $M(\mu) = \{\omega_1 = 0111, \omega_2 = 0101, \omega_3 = 0110, \omega_4 = 0100, \omega_5 = 0011, \omega_6 = 0001, \omega_7 = 0010, \omega_8 = 0000, \omega_9 = 1111, \omega_{10} = 1101, \omega_{11} = 1110, \omega_{12} = 1100\}$. It is clear that $\bigwedge_{S_i \in K_1 \cup K_2 \cup K_3} S_i \wedge \mu$ is consistent (the knowledge base S_i is viewed as a formula), i.e. ω_1 is its only model. Let $\mathbf{X} = \{X_1, X_2, X_3\}$, where $X_1 = X_2 = bo$ and $X_3 = d_H$. Let $k = 2$. We then have $M(\Delta_\mu^{2,\mathbf{X}}(E)) = \{\omega_1, \omega_9\}$. So $\Delta_\mu^{2,\mathbf{X}}(E) \not\equiv \bigwedge_{S_i \in K_1 \cup K_2 \cup K_3} S_i \wedge \mu$.

In Example 3, the original stratified knowledge bases are jointly consistent with μ. So intuitively, a possible world is a model of resulting knowledge base of merging if it is a model of every K_i $(i = 1, 2, 3)$ and μ. However, ω_9, which is a model of $\Delta_\mu^{2,\mathbf{X}}(E)$, is not a model of K_1 because it falsifies $\neg p$. This problem will be further discussed in Section 7.

We have the following refined definition of quota-based merging operators.

Definition 8. *Let $E = \{K_1, ..., K_n\}$ be a SKP, μ be a formula, and let k be an integer. Let $\mathbf{X} = (X_1, ..., X_n)$ be a set of ordering strategies, where X_i $(i = 1, ..., n)$ are ordering strategies attached to K_i. Suppose $<_{K_i, X_i}$ is the complete, transitive and asymmetric relation on 2^Ω obtained by K_i and X_i. The resulting knowledge base of refined k-quota merging operator, denoted by $\Delta_{r,\mu}^{k,\mathbf{X}}(E)$, is defined in a model-theoretic way as follows:*

$$M(\Delta_{r,\mu}^{k,\mathbf{X}}(E)) = \begin{cases} \{\omega \in M(\mu) | \forall K_i \in E \ \omega \models K_i\} & \textit{if not empty,} \\ \{\omega \in M(\mu) | \sharp(\{K_i \in E | \ \omega \in Min(2^\Omega, <_{K_i, X_i})\}) \geq k\} & \textit{otherwise} \end{cases}$$

Clearly, we have the following proposition.

Proposition 5. *Let $E = \{K_1, ..., K_n\}$ be a multi-set of stratified knowledge bases, μ be a formula, and let k be an integer. Let $\mathbf{X} = (X_1, ..., X_n)$ be a set of ordering strategies, where X_i $(i = 1, ..., n)$ are ordering strategies attached to K_i. We have $\Delta_{r,\mu}^{k,\mathbf{X}}(E) \vdash \Delta_\mu^{k,\mathbf{X}}(E)$.*

5.4 Flat case

In this section, we apply our merging operators to the classical knowledge bases. Since our merging operators are based on the ordering strategies, we need to consider the ordering strategies for classical knowledge bases.

Proposition 6. *Let K be a classical knowledge base. Suppose X is an ordering strategy, then*

1. *for $X = bo$ and $X = maxsat$, we have $\omega \preceq_X \omega'$ iff $\omega \models K$*
2. *for $X = leximin$, let $K(\omega) = \{\phi \in K : \omega \models \phi\}$, we have $\omega \preceq_X \omega'$ iff $|K(\omega)| \geq |K(\omega')|$*
3. *for $X = d$, we have $\omega \preceq_X \omega'$ iff $d(\omega, K) \leq d(\omega, K')$.*

By Proposition 6, the best out ordering and the maxsat ordering are reduced to the same ordering when knowledge base is flat. Furthermore, the leximin ordering can be used to order possible worlds when the knowledge base is inconsistent.

We have the following propositions.

Proposition 7. *Let $E = \{K_1, ..., K_n\}$ be a multi-set of knowledge bases, μ be a formula, and k be an integer. Suppose $X_i = bo$ or $maxsat$ for all i. Then*

$$\Delta_{r,\mu}^{k,\mathbf{X}}(E) \equiv \Delta_{\mu}^{k}(E).$$

Proposition 7 tells us that, in the flat case, the result of our refined quota-based merging operators is equivalent to that of the quota merging operators when the ordering strategies are the best out ordering or the maxsat ordering. By Proposition 1, 2, 4 and 7, we have the following result.

Proposition 8. *Let $E = \{K_1, ..., K_n\}$ be a multi-set of knowledge bases, μ be a formula, and k be an integer. Suppose $X_i = leximin$ or d, then*

$$\Delta_{r,\mu}^{k,\mathbf{X}}(E) \vdash \Delta_{\mu}^{k}(E),$$

but not vice verse.

Let us look at an example.

Example 4. Let $E = \{K_1, K_2\}$, where $K_1 = \{p_1 \vee p_2, p_3, \neg p_3\}$ and $K_2 = \{p_1, p_2, p_3\}$, $\mu = \{(p_1 \vee p_3) \wedge p_2\}$ and $k = 2$. So $Mod(\mu) = \{\omega_1 = 110, \omega_2 = 111, \omega_3 = 011\}$. Let $\mathbf{X} = (X_1, X_2)$, where $X_1 = leximin$ and $X_2 = bo$ are ordering strategies of K_1 and K_2 respectively. The computations are given in Table 3 below.

ω	K_1	K_2
110	1	2
111	1	1
011	1	2

Table 3

According to Table 3, $\omega_2 = 111$ is the only model which belong to the level 1 of the strata of both Ω_{K_1,X_1} and Ω_{K_2,X_2}. So $M(\Delta_{\mu}^{2,\mathbf{X}}(E)) = \{111\}$. However, if we apply the quota merging operator, since K_1 and K_2 are inconsistent, it is clear that $M(\Delta_{\mu}^{k}(E)) = \emptyset$.

6 Computational Complexity

We now discuss the complexity issue. First we need to consider the computational complexity of stratifying Ω from a stratified knowledge base. In [15], two important problems for logical preference representation languages were considered. We express them as follows.

Definition 9. *Given a stratified knowledge base K and two interpretations ω and ω', the COMPARISON problem consists of determining whether $\omega \preceq_X \omega'$, where X denotes an ordering strategy. The NON-DOMINANCE problem consists of determining whether ω is* non-dominated *for $\preceq_X$, that is, there is not ω' such that $\omega' \prec_X \omega$.*

It was shown in [15] that the NON-DOMINANCE problem is usually a hard problem, i.e **coNP**-complete. We have the following proposition on NON-DOMINANCE problem for ordering strategies in Section 3.

Proposition 9. *Let K be a stratified knowledge base. For $X = bo$, $maxsat$, or $lexmin$:*
(1) COMPARISON is in P*, where* P *denotes the class of problems decidable in deterministic polynomial time.*
*(2) NON-DOMINANCE is co**NP**-complete.*

To stratify Ω, we need to consider the problem *determining all non-dominated interpretations*, which is computational much harder than the NON-DOMINANCE problem. To simplify the computation of our merging operators, we assume that Ω is stratified from each stratified knowledge base during an off-line preprocessing stage.

Let Δ be a merging operator. The following decision problem is denoted as MERGE(Δ):

- **Input :** a 4-tuple $\langle E, \mu, \psi, \mathbf{X} \rangle$ where $E = \{K_1, ..., K_n\}$ is a SKP, μ is a formula, and ψ is a formula; $\mathbf{X} = (X_1, ..., X_n)$, where X_i is the ordering strategy attached to K_i.
- **Question :** Does $\Delta_\mu(E) \models \psi$ hold?

Proposition 10. *MERGE($\Delta^{k,\mathbf{X}}$) is Co***NP***-complete and MERGE($\Delta_r^{k,\mathbf{X}}$) is* BH (2)*-complete.*

The proof of Proposition 10 is similar to that of Proposition 4 in [9]. Proposition 10 shows that the complexities of both $\Delta^{k,\mathbf{X}}$ operators and $\Delta_r^{k,\mathbf{X}}$ operators are located at a low level of the boolean hierarchy. Furthermore, the computation of $\Delta^{k,\mathbf{X}}$ operators is easier than that of $\Delta_r^{k,\mathbf{X}}$ operators (under the usual assumptions of complexity theory).

7 Logical Properties

Many logical properties have been proposed to characterize a belief merging operator. We introduce the set of postulates proposed in [11], which is used to characterize Integrity Constraints (IC) merging operators.

Definition 10. *Let E, E_1, E_2 be knowledge profiles, K_1, K_2 be consistent knowledge bases, and μ, μ_1, μ_2 be formulas from $\mathcal{L}_{PS}$. Δ is an IC merging operator iff it satisfies the following postulates:*
(IC0) $\Delta_\mu(E) \models \mu$
(IC1) *If μ is consistent, then $\Delta_\mu(E)$ is consistent*
(IC2) *If $\bigwedge E$ is consistent with μ, then $\Delta_\mu(E) \equiv \bigwedge E \wedge \mu$, where $\bigwedge(E) = \wedge_{K_i \in E} K_i$*
(IC3) *If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$*
(IC4) *If $K_1 \models \mu$ and $K_2 \models \mu$, then $\Delta_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent iff $\Delta_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent*
(IC5) $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2) \models \Delta_\mu(E_1 \sqcup E_2)$
(IC6) *If $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$ is consistent, then $\Delta_\mu(E_1 \sqcup E_2) \models \Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$*
(IC7) $\Delta_{\mu_1}(E) \wedge \mu_2 \models \Delta_{\mu_1 \wedge \mu_2}(E)$
(IC8) *If $\Delta_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\Delta_{\mu_1 \wedge \mu_2}(E) \models \Delta_{\mu_1}(E) \wedge \mu_2$*

The postulates are used to characterize an IC merging operator in classical logic. Detailed explanation of the above postulates can be found in [11].

Some postulates in Definition 10 need to be modified if we consider merging postulates for stratified knowledge bases, i.e., $(IC2)$, $(IC3)$ should be modified as:
(IC2$'$) Let $\bigwedge E = \wedge_{K_i \in E} \wedge_{\phi_{ij} \in K_i} \phi_{ij}$. If $\bigwedge E$ is consistent with μ, then $\Delta_\mu(E) \equiv \bigwedge E \wedge \mu$
(IC3$'$) If $E_1 \equiv_s E_2$ and $\mu_1 \equiv \mu_2$, then $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$

$(IC3')$ is stronger than $(IC3)$ because the condition of equivalence between two knowledge profiles is generalized to the condition of equivalence between two SKPs. We do not generalize $(IC4)$, the fairness postulate, which is hard to be adapted in the prioritized case because a stratified knowledge base may be inconsistent and there is no unique consequence relation for a stratified knowledge base [3].

Proposition 11. *$\Delta^{k,\mathbf{X}}$ satisfies (IC0), (IC5), (IC7), (IC8). The other postulates are not satisfied in the general case. $\Delta_r^{k,\mathbf{X}}$ satisfies (IC0), (IC2), (IC5), (IC7), (IC8). The other postulates are not satisfied in the general case.*

(IC1) is not satisfied by both $\Delta^{k,\mathbf{X}}$ and $\Delta^{k,\mathbf{X}}$ because the result of merging may be inconsistent. $\Delta^{k,\mathbf{X}}$ and $\Delta_r^{k,\mathbf{X}}$ do not satisfy (IC3′) because some ordering strategies may be syntax sensitive. A difference between $\Delta^{k,\mathbf{X}}$ and $\Delta_r^{k,\mathbf{X}}$ is that $\Delta^{k,\mathbf{X}}$ does not satisfy the postulate (IC2′), whilst $\Delta_r^{k,\mathbf{X}}$ satisfies this postulate. The following proposition shows that when the ordering strategies are either best out ordering or maxsat ordering, then both operators satisfy (IC3′).

Proposition 12. *Suppose $X_i = bo$, maxsat, then $\Delta^{k,\mathbf{X}}$ satisfies (IC0), (IC2), (IC3′), (IC5), (IC7), (IC8). The other postulates are not satisfied in the general case.*

8 Related Work

Merging of stratified knowledge bases is often handled in the framework of possibilistic logic [8] or ordinal conditional function [20]. In possibilistic logic, the merging problems are often solved by aggregating *possibility distributions*, which are mappings from Ω to a common scale such as [0,1], using some *combination modes*. Then the syntactic counterpart of these combination modes can be defined accordingly [4, 5]. In [7], the merging is conducted by merging *epistemic states* which are (total) functions from the set of interpretations to $\mathbf{N}$, the set of natural numbers. We now discuss two main differences between our merging operators and previous merging operators for stratified knowledge bases.

First, our operators are semantically defined in a model-theoretic way and others are semantically defined by distribution functions such as possibility distributions in possibilistic logic framework. In the flat case, our merging operators belong to model-based merging operators in classical logic, so it is independent of syntactical form of the knowledge bases. In contrast, other merging operators are usually syntax-based ones in the flat case.

Second, most of previous merging operators are based on the commensurability assumption, that is, all agents use a common scale to rank their beliefs. In [4], a merging approach for stratified knowledge base is proposed which drops the commensurability assumption. However, their approach is based on the assumption that there is an ordering relation between two stratified knowledge bases K_1 and K_2, i.e. K_1 has priority over K_2. In contrast, our merging operators do not require any of above assumptions and are flexible enough to merge knowledge bases which are stratified by a total pre-ordering on their elements.

In [18], we proposed a family of lexicographic merging operators for stratified knowledge bases. Our quota-based merging operators only use the most preferred possible worlds *w.r.t* each ordering strategy. That is, suppose $\Omega_{K,X} = (\Omega_1, ..., \Omega_m)$, then only Ω_1 is used to define the quota-based operators. Whilst the lexicographic merging operators utilize the rest of the structure of $\Omega_{K,X}$. Therefore, the lexicographic merging operators are refinement of the quota-based operators. However, this refinement is paid by higher computational complexity.

9 Conclusions

In this paper, we have proposed a family of quota-based operators to merge stratified knowledge bases under integrity constraints. Our operators generalize the quota merging operators for classical knowledge bases. The computational complexity of our merging operators has been analyzed. Under an additional assumption, the complexities of both $\Delta^{k,\mathbf{X}}$ operators and $\Delta_r^{k,\mathbf{X}}$ operators are located at a low level of the boolean hierarchy. Furthermore, the computation of $\Delta^{k,\mathbf{X}}$ operators is easier than that of $\Delta_r^{k,\mathbf{X}}$ operators (under the usual assumptions of complexity theory). Finally, we have generalized the set of postulates defined in [11] and shown that our operators satisfy most of the generalized postulates.

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