A Split-combination Method for Merging Inconsistent Possibilistic Knowledge Bases

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Abstract

In this paper, a new method for merging multiple inconsistent knowledge bases in the framework of possibilistic logic is presented. We divide the fusion process into two steps: one is called the splitting step and the other is called the combination step. Given several inconsistent possibilistic knowledge bases (i.e. the union of these possibilistic bases is inconsistent), we split each of them into two subbases according to the *upper free degree* of their union, such that one subbase contains formulas whose necessity degrees are less than the *upper free degree* and the other contains formulas whose necessity degrees are greater than the *upper free degree*. In the second step, we combine the former using the maximum (or more generally, *T-conorm*) combination mode, while combining the latter using the minimum (or more generally, *T-norm*) combination mode. The union of the possibilistic bases obtained by the second step is taken as the final result of the combination of the possibilistic bases that we want to merge. We prove that when the possibilistic bases are consistent with each other, the result of our new combination method is equivalent to that of the minimum (*T-norm*) based combination mode. However, when the sources are inconsistent with each other, the result of our combination mode is better than that obtained by using the maximum (*T-conorm*) based mode. An alternative approach to splitting the possibilistic bases is introduced in the last section. The combination mode obtained by this splitting method can be applied to combine knowledge bases which are flat, i.e., without any priority between their elements.

Introduction

In many cases, we confront the problem of merging inconsistent information from different sources (Abidi and Gonzalez 1992; Cholvy 1992; Cholvy and Hunter 1997; Baral et al. 1992; Gärdenfors 1988; Liberatore and Schaerf 1998; Konieczny 2000; Konieczny and Pino Pérez 1998; 2002; Dubois, Lang, and Prade 1992; Benferhat et al 1997a; 1998; 1999; 2002; William 1994; 1996). Possibilistic logic (Dubois, Lang, and Prade 1994) provides a good framework to deal with fusion problems when information is pervaded with uncertainty and inconsistency (Dubois, Lang, and Prade 1992; Benferhat et al. 1997a; 1998; 2002). In (Benferhat, Dubois, and Prade 1997a), some syntactic combination modes for merging uncertain propositional knowledge bases, in the framework of possibilistic logic, are proposed. These modes are the counterparts of the semantic combination modes which are applied to possibility distributions. Among them, two operators, *maximum* (or more generally, *T-conorm*) and *minimum* (or more generally, *T-norm*), are used to combine consistent and inconsistent sources of information respectively. Given two *possibilistic knowledge bases* $\mathcal{B}_1 = \{(\phi_i,\ \alpha_i),\ i = 1, ..., n\}$ and $\mathcal{B}_2 = \{(\psi_j,\ \beta_j), j = 1, ..., m\}$, where ϕ_i and ψ_j are classical propositional formulas, and α_i and β_j belonging to [0,1] are necessity degrees of ϕ_i and ψ_j respectively, the syntactic results of merging $\mathcal{B}_1$ and $\mathcal{B}_2$ by the maximum combination mode and the minimum combination mode are $\mathcal{B}_{dm} = \{(\phi_i \vee \psi_j,\ min(\alpha_i, \beta_j)) | (\phi_i,\ \alpha_i) \in \mathcal{B}_1,\ (\psi_j,\ \beta_j) \in \mathcal{B}_2\}$ and $\mathcal{B}_{cm} = \mathcal{B}_1 \cup \mathcal{B}_2$ respectively. $\mathcal{B}_{dm}$ is always consistent provided that $\mathcal{B}_1$ or $\mathcal{B}_2$ is consistent, whilst $\mathcal{B}_{cm}$ is consistent only if the union of $\mathcal{B}_1$ and $\mathcal{B}_2$ is consistent. So the maximum combination mode is more advisable than the minimum combination mode to deal with inconsistency. However, when the union of $\mathcal{B}_1$ and $\mathcal{B}_2$ is consistent, the minimum combination mode will result in a more *specific* possibilistic knowledge base. That is, the possibility distribution of the combination of $\mathcal{B}_1$ and $\mathcal{B}_2$ by the minimum combination mode is more *specific* than that of the combination of $\mathcal{B}_1$ and $\mathcal{B}_2$ by the maximum combination mode. Therefore, the maximum combination mode is too *cautious* to be used to merge consistent possibilistic knowledge bases.

In this paper, we propose a split-combination method based on the maximum (or T-conorm) and the minimum (or T-norm) operators. Given two possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$ (where $\mathcal{B}_1 \cup \mathcal{B}_2$ is inconsistent but each of them is individually consistent), we first split each of them into two subbases such that $\mathcal{B}_1 = \mathcal{C}_1 \cup \mathcal{D}_1$ and $\mathcal{B}_2 = \mathcal{C}_2 \cup \mathcal{D}_2$ according to the *upper free degree* of $\mathcal{B}_1 \cup \mathcal{B}_2$. The *upper free degree* of a possibilistic knowledge base $\mathcal{B}$ is the minimum number α in [0,1] such that the *strict α-cut* of $\mathcal{B}$ does not contain any conflict formulas. $\mathcal{C}_1 \cup \mathcal{C}_2$ is the inconsistent part of $\mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{D}_1 \cup \mathcal{D}_2$ is the consistent part of $\mathcal{B}_1 \cup \mathcal{B}_2$. In the second step, we combine $\mathcal{C}_1$ and $\mathcal{C}_2$ using the maximum (or more general, T-conorm) combination mode, while combining $\mathcal{D}_1$ and $\mathcal{D}_2$ using the minimum (or more generally, T-norm) combination mode. Finally, the union of the possibilistic bases obtained by the second step is taken as the result of the combination of $\mathcal{B}_1$ and $\mathcal{B}_2$. We prove that the new combination mode reduces to the minimum combination mode when no conflict exists and that it is better than the maximum combination mode for merging inconsistent knowledge bases.

The new combination method cannot be applied to merge flat (or classical) knowledge bases, i.e., knowledge bases without any priorities between their elements, because the *upper free degree* used to split the possibilistic bases is related to priority. Therefore we propose an alternative approach to split knowledge bases which do not involve priority. The revised split-combination method can then be applied to merging classical knowledge bases.

This paper is organized as follows. In Section 2, we introduce our split-combination method based on the maximum and the minimum operators for merging inconsistent knowledge bases, in the framework of possibilistic logic. We will discuss the properties of the new method in this section. Then in Section 3, a semantic interpretation of the method is presented. In Section 4, the maximum and the minimum operators are extended to the T-norm and the T-conorm operators, and the corresponding properties are investigated. An alternative approach to splitting the possibilistic knowledge bases is proposed in Section 5. Finally, in Section 6, we summarize the paper.

A Split-combination based Merging Method

Some basic definitions

In this section, we introduce some basic definitions in possibilistic logic (Dubois, Lang, and Prade 1994) and define an *upper free degree* that will be used to split possibilistic bases. We only consider a finite propositional language denoted by $\mathcal{L}$. The classical consequence relation is denoted as $\models$. ϕ, ψ, γ,... represent classical formulas.

In possibilistic logic, at the semantic level, the basic notion is a *possibility distribution*, denoted by π, which is a mapping from a set of interpretations Ω to the interval [0,1]. $\pi(\omega)$ represents the possibility degree of the interpretation ω with the available beliefs. From a *possibility distribution* π, two measures defined on a set of propositional or first order formulas can be determined. One is the possibility degree of formula ϕ, denoted as $\Pi(\phi) = max\{\pi(\omega) : \omega \models \phi\}$. The other is the necessity degree of formula ϕ, and is defined as $N(\phi) = 1 - \Pi(\neg\phi)$.

At the syntactic level, a formula, called a *possibilistic formula*, is represented by a pair $(\phi,\ \alpha)$ where ϕ is a formula and $\alpha \in [0, 1]$. Then uncertain pieces of information can be represented by a *possibilistic knowledge base* which is a finite set of *possibilistic formulas* of the form $\mathcal{B} = \{(\phi_i,\ \alpha_i) : i = 1, ..., n\}$. The *possibilistic formula* $(\phi_i,\ \alpha_i)$ means that the necessity degree of ϕ_i is at least equal to α_i, i.e. $N(\phi_i) \geq \alpha_i$. In this paper, we only consider possibilistic knowledge bases where every formula ϕ is a classical propositional formula. The classical base associated with $\mathcal{B}$ is denoted as $\mathcal{B}^*$, namely $\mathcal{B}^* = \{\phi_i | (\phi_i,\ \alpha_i) \in \mathcal{B}\}$. A *possibilistic base* $\mathcal{B}$ is consistent if and only if its classical base $\mathcal{B}^*$ is consistent.

Given a *possibilistic base* $\mathcal{B}$, a unique *possibility distribution*, denoted by $\pi_{\mathcal{B}}$ can be obtained by the principle of minimum specificity. For all $\omega \in \Omega$,

$$\pi_{\mathcal{B}}(\omega) = \begin{cases} 1 & \text{if } \forall(\phi_i,\ \alpha_i) \in \mathcal{B}, \omega \models \phi_i, \\ 1 - max\{\alpha_i | \omega \not\models \phi_i\} & \text{otherwise.} \end{cases} \quad (1)$$

The *inconsistency degree* of $\mathcal{B}$, which defines the level of inconsistency of $\mathcal{B}$, is defined by

$$Inc(\mathcal{B}) = 1 - max_{\omega} \pi_{\mathcal{B}}(\omega).$$

Definition 1 (Dubois, Lang, and Prade 1994) *Let $\mathcal{B}$ be a possibilistic base, and $\alpha \in [0, 1]$. We call the α-cut (respectively strict α-cut) of $\mathcal{B}$, denoted by $\mathcal{B}_{\geq\alpha}$ (respectively $\mathcal{B}_{>\alpha}$), the set of classical formulas in $\mathcal{B}$ having a necessity degree at least equal to α (respectively strictly greater than α).*

The *inconsistency degree* of $\mathcal{B}$ in terms of the α-cuts can be equivalently defined as (Dubois, Lang, and Prade 1994):

$$Inc(\mathcal{B}) = max\{\alpha_i | \mathcal{B}_{\geq\alpha_i}\ is\ inconsistent\}.$$

Definition 2 (Dubois, Lang, and Prade 1994) *Let $\mathcal{B}$ be a possibilistic base. Let $(\phi,\ \alpha)$ be a piece of information with $\alpha > Inc(\mathcal{B})$. $(\phi,\ \alpha)$ is said to be a consequence of $\mathcal{B}$, denoted by $\mathcal{B} \vdash_\pi (\phi,\ \alpha)$, iff $\mathcal{B}_{\geq\alpha} \vdash \phi$.*

Definition 3 (Benferhat, Dubois, and Prade 1997b) *A subbase $\mathcal{A}$ of $\mathcal{B}$ is said to be minimally inconsistent if and only if it satisfies the following two requirements:*

- *$(\mathcal{A})^* \models false$, where $(\mathcal{A})^*$ is the classical base of $\mathcal{A}$, and*
- *$\forall\phi \in (\mathcal{A})^*$, $(\mathcal{A})^* - \{\phi\} \not\models false$.*

Definition 4 (Benferhat, Dubois, and Prade 1997b) *A possibilistic formula $(\phi,\ \alpha)$ is said to be $free$ in $\mathcal{B}$ iff it does not belong to any minimally inconsistent subbase of $\mathcal{B}$. Free$(\mathcal{B})$ denotes the set of free formulas in $\mathcal{B}$.*

Another concept that is based on the *minimally inconsistent subbase* and is related with the free formula is defined as follows.

Definition 5 *A possibilistic formula $(\phi,\ \alpha)$ is said to be in conflict in $\mathcal{B}$ iff it belongs to some minimally inconsistent subbase of $\mathcal{B}$.*

Clearly, a formula is a conflict formulas of $\mathcal{B}$ iff it is not a free formula of $\mathcal{B}$.

Now, we give our definition of the *upper free degree* of a possibilistic base, based on the definition of *conflict formulas*.

Definition 6 *The upper free degree of a possibilistic base $\mathcal{B}$ is defined as:*

$$Free_{upp}(\mathcal{B}) = min\{\alpha_i \in [0, 1] : \mathcal{B}_{>\alpha_i}\ does\ not\ contain\ any\ conflict\ formulas\} \quad (2)$$

$Free_{upp}(\mathcal{B}) = 0$ when $\mathcal{B}$ is consistent. $\mathcal{B}_{>\alpha_i}$ contains some free formulas of $\mathcal{B}$, but not all of them.

Definition 7 *(upper-free-degree-based splitting) Given a possibilistic base $\mathcal{B}$, the splitting of $\mathcal{B}$ with regard to $Free_{upp}(\mathcal{B})$ is defined as a pair $< \mathcal{C}, \mathcal{D} >$ such that $\mathcal{B} = \mathcal{C} \cup \mathcal{D}$, where*

$$\mathcal{C} = \{(\phi,\ \alpha) \in \mathcal{B} \mid \alpha \leq Free_{upp}(\mathcal{B})\}$$

and

$$\mathcal{D} = \{(\phi,\ \alpha) \in \mathcal{B} \mid \alpha > Free_{upp}(\mathcal{B})\}.$$

By Definition 6, $\mathcal{C}$ is inconsistent if $Free_{upp}(\mathcal{B}) > 0$ and $\mathcal{D}$ is always consistent. It is clear that $Free_{upp}(\mathcal{B}) \leq Inc(\mathcal{B})$, for each possibilistic knowledge base $\mathcal{B}$.

Let us look at an example to illustrate how to split a possibilistic base.

Example 1 *Given a possibilistic knowledge base $\mathcal{B} = \{(\phi,\ 0.4), (\neg\phi \vee \psi,\ 0,3), (\neg\phi \vee \neg\gamma,\ 0.6), (\neg\psi \vee \gamma,\ 0.5), (\neg\psi \vee \delta,\ 0.9), (\phi \vee \delta,\ 0.7)\}$, by Definition 6, the upper free degree of $\mathcal{B}$ is 0.6. $\mathcal{B}$ is then split into $< \mathcal{C}, \mathcal{D} >$ such that*

$$\begin{aligned} \mathcal{C} &= \{(\phi,\ 0.4), (\neg\phi \vee \psi,\ 0,3), (\neg\phi \vee \neg\gamma,\ 0.6), \\ &\quad (\neg\psi \vee \gamma,\ 0.5)\}, \\ \mathcal{D} &= \{(\neg\psi \vee \delta,\ 0.9), (\phi \vee \delta,\ 0.7)\}. \end{aligned}$$

In (Benferhat, Dubois, and Prade 1997a), some combination rules for merging possibilistic bases are proposed. Among them, two basic combination modes, the maximum combination mode and the minimum combination mode, are introduced to merge inconsistent and consistent sources of information respectively. Given two possibilistic bases $\mathcal{B}_1$ and $\mathcal{B}_2$ with possibility distributions $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ respectively, the semantic results of the combination of $\mathcal{B}_1$ and $\mathcal{B}_2$ using the maximum combination mode and the minimum combination mode are

$$\forall w, \pi_{\mathcal{B}_{dm}}(w) = max\{\pi_{\mathcal{B}_1}(w),\ \pi_{\mathcal{B}_2}(w)\}, \tag{3}$$

$$\forall w, \pi_{\mathcal{B}_{cm}}(w) = min\{\pi_{\mathcal{B}_1}(w),\ \pi_{\mathcal{B}_2}(w)\} \tag{4}$$

respectively. And the syntactic results which are the counterpart of the semantic results are

$$\begin{aligned} \mathcal{B}_{dm} &= \{(\phi_i \vee \psi_j,\ min(\alpha_i,\ \beta_j)) | (\phi_i,\ \alpha_i) \in \mathcal{B}_1, \\ &\quad and\ (\psi_j,\ \beta_j) \in \mathcal{B}_2\}, \end{aligned} \tag{5}$$

$$\mathcal{B}_{cm} = \mathcal{B}_1 \cup \mathcal{B}_2. \tag{6}$$

B_{dm} and B_{cm} are referred to as the results of the *disjunctive* and *conjunctive* combination respectively (Benferhat, Dubois, and Prade 1997a).

A split-combination method

Based on Definition 7, we now introduce our method which splits and combines two possibilistic bases $\mathcal{B}_1$ and $\mathcal{B}_2$, where $\mathcal{B}_1 \cup \mathcal{B}_2$ is inconsistent but each of them is individually consistent. The procedure consists of the following steps:

Upper-free-degree-based-split-combination(S-C Combination) Algorithm

- **Step 0:** Let $\mathcal{B}_1 = \{(\phi_1,\ \alpha_1), ..., (\phi_n,\ \alpha_n)\}$ and $\mathcal{B}_2 = \{(\psi_1,\ \beta_1), ..., (\psi_m,\ \beta_m)\}$ be two possibilistic bases, compute $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$.
- **Step 1:** Split $\mathcal{B}_1$ and $\mathcal{B}_2$ with regard to $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$ as follows. Let $< \mathcal{C}', \mathcal{D}' >$ be a splitting of $\mathcal{B}_1 \cup \mathcal{B}_2$ with regard to $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$. The splitting of $\mathcal{B}_1$ is a pair $< \mathcal{C}_1, \mathcal{D}_1 >$ such that $\mathcal{C}_1 = \mathcal{C}' \cap \mathcal{B}_1$ and $\mathcal{D}_1 = \mathcal{D}' \cap \mathcal{B}_1$. The splitting of $\mathcal{B}_2$ is a pair $< \mathcal{C}_2, \mathcal{D}_2 >$ such that $\mathcal{C}_2 = \mathcal{C}' \cap \mathcal{B}_2$ and $\mathcal{D}_2 = \mathcal{D}' \cap \mathcal{B}_2$.
- **Step 2:** Combine $\mathcal{C}_1$ and $\mathcal{C}_2$ by the maximum operator and combine $\mathcal{D}_1$ and $\mathcal{D}_2$ by the minimum operator, as shown by Equation (5) and Equation (6), the results are

$$\begin{aligned} \mathcal{C} &= \{(\phi_i \vee \psi_j,\ min(\alpha_i,\ \beta_j)) | (\phi_i,\ \alpha_i) \in \mathcal{C}_1, \\ &\quad (\psi_j,\ \beta_j) \in \mathcal{C}_2\}, \end{aligned} \tag{7}$$

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2. \tag{8}$$

- **Step 3:** The final result of the S-C combination method, denoted by $\mathcal{B}_{S-C}$, is $\mathcal{C} \cup \mathcal{D}$.

In Step 1 above, we split $\mathcal{B}_1$ into $\mathcal{C}_1$ and $\mathcal{D}_1$ and split $\mathcal{B}_2$ into $\mathcal{C}_2$ and $\mathcal{D}_2$ by the *upper free degree* of $\mathcal{B}_1 \cup \mathcal{B}_2$. By Definition 6 and Definition 7, all the conflict formulas in $\mathcal{B}_1 \cup \mathcal{B}_2$ occur in $\mathcal{C}_1 \cup \mathcal{C}_2$. So $\mathcal{C}_1 \cup \mathcal{C}_2$ is inconsistent if $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2) > 0$ and $\mathcal{D}_1 \cup \mathcal{D}_2$ is consistent. Since the maximum operator is more advisable for combining inconsistent sources and the minimum operator is more advisable combining consistent sources, we combine $\mathcal{C}_1$ and $\mathcal{C}_2$ by the maximum operator, and combine $\mathcal{D}_1$ and $\mathcal{D}_2$ by the minimum operator in Step 2. We give an example to illustrate the algorithm.

Example 2 *Given two possibilistic knowledge bases $\mathcal{B}_1 = \{(\phi,\ 0.2), (\psi,\ 0.4), (\gamma,\ 0.7)\}$ and $\mathcal{B}_2 = \{(\phi,\ 0.6), (\neg\psi,\ 0.5),\ (\gamma,\ 0.3)\}$. The upper free degree of $\mathcal{B}_1 \cup \mathcal{B}_2$ is 0.5, so $\mathcal{B}_1$ and $\mathcal{B}_2$ are split as $< \mathcal{C}_1, \mathcal{D}_1 >$ and $< \mathcal{C}_2, \mathcal{D}_2 >$ such that*

$$\mathcal{C}_1 = \{(\phi,\ 0.2), (\psi,\ 0.4)\}, \ \ \mathcal{D}_1 = \{(\gamma,\ 0.7)\}$$

and

$$\mathcal{C}_2 = \{(\neg\psi,\ 0.5), (\gamma,\ 0.3)\}, \ \ \mathcal{D}_2 = \{(\phi,\ 0.6)\}.$$

Applying the maximum combination mode to $\mathcal{C}_1$ and $\mathcal{C}_2$, and applying the minimum combination mode to $\mathcal{D}_1$ and $\mathcal{D}_2$, we get $\mathcal{C} = \{(\phi \vee \gamma,\ 0.2), (\phi \vee \neg\psi,\ 0.2), (\psi \vee \gamma,\ 0.3)\}$ and $\mathcal{D} = \{(\gamma,\ 0.7), (\phi,\ 0.6)\}$. The final combination result of $\mathcal{B}_1$ and $\mathcal{B}_2$ is

$$\begin{aligned} \mathcal{B}_{S-C} &= \{(\phi \vee \gamma,\ 0.2), (\phi \vee \neg\psi,\ 0.2), (\psi \vee \gamma,\ 0.3), \\ &\quad (\gamma,\ 0.7), (\phi,\ 0.6)\}. \end{aligned}$$

Properties of the new combination method

For two possibilistic bases $\mathcal{B}_1$ and $\mathcal{B}_2$, if $\mathcal{B}_1 \cup \mathcal{B}_2$ is consistent, by Definition 6, we have $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2) = 0$. When we split $\mathcal{B}_1$ and $\mathcal{B}_2$ using $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, we obtain $\mathcal{C}_1 = \emptyset$, $\mathcal{D}_1 = \mathcal{B}_1$ and $\mathcal{C}_2 = \emptyset$, $\mathcal{D}_2 = \mathcal{B}_2$, which results in $\mathcal{B}_{S-C} = \mathcal{B}_1 \cup \mathcal{B}_2$. Therefore, the S-C combination mode is equivalent to the minimum combination mode when sources are consistent. Next we give some properties of the new combination method when $\mathcal{B}_1 \cup \mathcal{B}_2$ is inconsistent. In the following, we always assume that the original possibilistic knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$ are individually consistent.

Proposition 1 *Possibilistic base $\mathcal{C}$ obtained by Step 2 of the S-C algorithm is consistent if initial knowledge bases are individually consistent.*

Proof. *Since $\mathcal{C}_1$ is consistent, there exists a model of it. Assume υ is a valuation such that $\upsilon(\phi_i) = true$ for every formula $\phi_i \in (\mathcal{C}_1)^*$, then since each formula in $\mathcal{C}$ has the form $\phi_i \vee \psi_j$, we must have $\upsilon(\phi_i \vee \psi_j) = true$, for all $\phi_i \vee \psi_j \in (\mathcal{C})^*$. Therefore, the valuation υ is also a model of $\mathcal{C}$, thus $\mathcal{C}$ is consistent.*

Proposition 2 *The final possibilistic base $\mathcal{B}_{S-C}$ obtained by Step 3 in the S-C algorithm is consistent.*

Proof. *Suppose $\mathcal{B}_{S-C}$ is inconsistent, then we have $(\mathcal{C} \cup \mathcal{D})^* \models false$. By Equation (7), we have $(\mathcal{C}_1)^* \models (\mathcal{C})^*$. Therefore $(\mathcal{C}_1 \cup \mathcal{D})^* \models false$. However we have assumed that $\mathcal{B}_1$ is consistent, so $\mathcal{C}_1$ must be consistent. Therefore there must exist some formulas in $\mathcal{D}$ which are in conflict with formulas in $\mathcal{C}$. This is a contradiction, because all formulas in $\mathcal{D}$ are free in $\mathcal{B}_1 \cup \mathcal{B}_2$. This completes our proof.*

Proposition 3 *Given two possibilistic bases $\mathcal{B}_1$ and $\mathcal{B}_2$, let $\mathcal{B}_{S-C}$ be the possibilistic base obtained by the S-C combination mode and $\mathcal{B}_{dm}$ be the possibilistic base obtained by the maximum combination mode, then*

$$\mathcal{B}_{S-C} \vdash_\pi (\phi,\ \alpha), \quad \textit{for all } (\phi,\ \alpha) \in \mathcal{B}_{dm} \tag{9}$$

Proof. *By Equation (5), every formula in $\mathcal{B}_{dm}$ has the form $(\phi_i \vee \psi_j,\ min(\alpha_i,\ \beta_j))$, where $(\phi_i,\ \alpha) \in \mathcal{B}_1$ and $(\psi_j,\ \beta_j) \in \mathcal{B}_2$, so we consider four cases:*

Case 1: *$\alpha_i,\ \beta_j \leq Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, then we have $(\phi_i \vee \psi_j,\ min(\alpha_i,\ \beta_j)) \in \mathcal{B}_{S-C}$. So $\mathcal{B}_{S-C} \vdash_\pi (\phi_i \vee \psi_j,\ min(\alpha_i,\ \beta_j))$.*

Case 2: *$\alpha_i > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, and $\beta_j \leq Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, then $min(\alpha_i,\ \beta_j) = \beta_j$. Since $\phi_i \models (\phi_i \vee \psi_j)$ and $\alpha_i \geq \beta_j$, we have $\mathcal{B}_{S-C} \vdash_\pi (\phi_i \vee \psi_j,\ min(\alpha_i,\ \beta_j))$.*

Case 3: *$\alpha_i \leq Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, and $\beta_j > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, this case is a dual to case 2.*

Case 4: *$\alpha_i > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$ and $\beta_j > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, we can suppose $\alpha_i > \beta_j$, then $min(\alpha_i,\ \beta_j) = \beta_j$. Since $\phi_i \models (\phi_i \vee \psi_j)$, we have $\mathcal{B}_{S-C} \vdash_\pi (\phi_i \vee \psi_j,\ min(\alpha_i,\ \beta_j))$.*

Thus we complete the proof.

The converse of Proposition 3 is false. Let us look at an counter-example.

Example 3 *Given two possibilistic bases $\mathcal{B}_1 = \{(\phi,\ 0.5), (\neg\phi \vee \psi,\ 0.6), (\neg\phi \vee \gamma,\ 0.8)\}$ and $\mathcal{B}_2 = \{(\neg\phi \vee \neg\psi,\ 0.4), (\psi,\ 0.3), (\phi \vee \delta,\ 0.9), (\neg\psi \vee \delta,\ 0.7), (\delta \vee \gamma,\ 0.5)\}$, the upper free degree of $\mathcal{B}_1 \cup \mathcal{B}_2$ is 0.6. Therefore we split $\mathcal{B}_1$ and $\mathcal{B}_2$ as $< \mathcal{C}_1, \mathcal{D}_1 >$ and $< \mathcal{C}_2, \mathcal{D}_2 >$ such that*

$$\mathcal{C}_1 = \{(\phi,\ 0.5), (\neg\phi \vee \psi,\ 0.6)\},\ \ \mathcal{D}_1 = \{(\neg\phi \vee \gamma,\ 0.8)\},$$

and

$$\mathcal{C}_2 = \{(\neg\phi \vee \neg\psi,\ 0.4), (\psi,\ 0.3), (\delta \vee \gamma,\ 0.5)\},$$

$$\mathcal{D}_2 = \{(\phi \vee \delta,\ 0.9), (\neg\psi \vee \delta,\ 0.7)\}.$$

Combining $\mathcal{C}_1$ and $\mathcal{C}_2$, and combining $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively gives

$$\begin{aligned}\mathcal{C} &= \{(\phi \vee \psi,\ 0.3), (\neg\phi \vee \psi,\ 0.3), (\phi \vee \delta \vee \gamma,\ 0.5),\\ &\quad (\neg\phi \vee \psi \vee \delta \vee \gamma,\ 0.5)\},\end{aligned}$$

$$\mathcal{D} = \{(\neg\phi \vee \gamma,\ 0.8), (\phi \vee \delta,\ 0.9), (\neg\psi \vee \delta,\ 0.7)\}.$$

So we have

$$\begin{aligned}\mathcal{B}_{S-C} &= \{(\phi \vee \psi,\ 0.3), (\neg\phi \vee \psi,\ 0.3), (\neg\phi \vee \gamma,\ 0.8),\\ &\quad (\neg\phi \vee \psi \vee \delta \vee \gamma,\ 0.5), (\phi \vee \delta \vee \gamma,\ 0.5),\\ &\quad (\phi \vee \delta,\ 0.9), (\neg\psi \vee \delta,\ 0.7)\}.\end{aligned}$$

If we combine $\mathcal{B}_1$ and $\mathcal{B}_2$ by the maximum combination mode, the result is

$$\begin{aligned}\mathcal{B}_{dm} &= \{(\neg\phi \vee \neg\psi \vee \delta \vee \gamma,\ 0.7), (\phi \vee \neg\psi \vee \delta,\ 0.5),\\ &\quad (\neg\phi \vee \psi \vee \delta \vee \gamma,\ 0.5), (\neg\phi \vee \psi \vee \gamma,\ 0.3),\\ &\quad (\phi \vee \psi,\ 0.3), (\phi \vee \delta,\ 0.5), (\neg\phi \vee \delta \vee \gamma,\ 0.5),\\ &\quad (\neg\phi \vee \psi,\ 0.3), (\neg\phi \vee \neg\psi \vee \gamma,\ 0.4),\\ &\quad (\phi \vee \delta \vee \gamma,\ 0.5)\}\end{aligned}$$

It is easy to check that all the possibilistic formulas in $\mathcal{B}_{dm}$ can be inferred from $\mathcal{B}_{S-C}$.

Proposition 3 and Example 3 show that the *F-C* combination mode is better than the *maximum* combination mode.

The proposed *F-C* combination algorithm is computationally very expensive. We are working on a method to reduce the computational complexity by viewing $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ as a layered knowledge base with the inconsistency degree as $Inc(\mathcal{B})$. When computing $Free_{upp}(\mathcal{B})$, only those formulae that have the necessity degrees greater than $Inc(\mathcal{B})$ need to be considered.

Semantic Aspects of The S-C Combination Method

In this section, we provide a semantic analysis of the S-C combination method. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two possibilistic bases, based on Step 1 of the S-C algorithm, they can be split as $\mathcal{B}_1 = \mathcal{C}_1 \cup \mathcal{D}_1$ and $\mathcal{B}_2 = \mathcal{C}_2 \cup \mathcal{D}_2$. Suppose $\pi_{\mathcal{C}_1}$, $\pi_{\mathcal{C}_2}$, $\pi_{\mathcal{D}_1}$ and $\pi_{\mathcal{D}_2}$ are the possibility distributions of $\mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{D}_1,\ \mathcal{D}_2$ respectively. Then the combination of $\mathcal{C}_1$ and $\mathcal{C}_2$ by the maximum combination mode is $\mathcal{C}$ such that

$$\forall\omega,\ \pi_{\mathcal{C}}(\omega) = max(\pi_{\mathcal{C}_1}(\omega),\ \pi_{\mathcal{C}_2}(\omega)),$$

and the combination of $\mathcal{D}_1$ and $\mathcal{D}_2$ by the minimum combination mode is $\mathcal{D}$ such that

$$\forall\omega,\ \pi_{\mathcal{D}}(\omega) = min(\pi_{\mathcal{D}_1}(\omega),\ \pi_{\mathcal{D}_2}(\omega)).$$

Lemma 1 *Let $\mathcal{B}_{S-C}$ be the combination result of the S-C method, then*

$$\begin{aligned}\pi_{\mathcal{B}_{S-C}}(w) &= min(max(\pi_{\mathcal{C}_1}(\omega),\ \pi_{\mathcal{C}_2}(\omega)),\\ &\quad min(\pi_{\mathcal{D}_1}(\omega), \pi_{\mathcal{D}_2}(\omega))).\end{aligned} \tag{10}$$

Proof. *Since $\mathcal{B}_{S-C} = \mathcal{C} \cup \mathcal{D}$, if ω is a model of $\mathcal{B}_{S-C}$, then it is a model of both $\mathcal{C}$ and $\mathcal{D}$, so $\pi_{\mathcal{B}_{S-C}}(w) = 1 = min(\pi_{\mathcal{C}}(\omega), \pi_{\mathcal{D}}(\omega)) = min(max(\pi_{\mathcal{C}_1}(\omega),\ \pi_{\mathcal{C}_2}(\omega)),\ min(\pi_{\mathcal{D}_1}(\omega),\ \pi_{\mathcal{D}_2}(\omega)))$. Otherwise, we have*

$$\begin{aligned}\pi_{\mathcal{B}_{S-C}}(\omega) &= \pi_{\mathcal{C} \cup \mathcal{D}}(\omega)\\ &= min(1 - \alpha_i | \omega \not\models \phi_i, (\phi_i,\ \alpha_i) \in \mathcal{C} \cup \mathcal{D})\\ &= min(\pi_{\mathcal{C}}(\omega),\ \pi_{\mathcal{D}}(\omega))\\ &= min(max(\pi_{\mathcal{C}_1}(\omega),\ \pi_{\mathcal{C}_2}(\omega)),\\ &\quad min(\pi_{\mathcal{D}_1}(\omega),\ \pi_{\mathcal{D}_2}(\omega))).\end{aligned}$$

Lemma 1 shows that the combination of two inconsistent knowledge bases at the semantic level can be equally split into two stages. In the first stage, the inconsistent and consistent formulae are merged separately using the *maximum* and the *minumum* operators respectively. In the second stage, the merged two subsets of formulae are combined again using the *minimum* operator. The operations performed in these two stages are symmetric to the operations in the syntactic combination procedure.

By the distributive law of *max* and *min*, that is, *min(a,max(b,c))=max(min(a,b),min(a,c))*, Equation (10) is equivalent to

$$\begin{aligned}\pi_{\mathcal{B}_{S-C}}(\omega) &= max(min(\pi_{\mathcal{C}_1}(\omega),\pi_{\mathcal{D}_1}(\omega),\pi_{\mathcal{D}_2}(\omega)),\\ &\quad min(\pi_{\mathcal{C}_2}(\omega),\pi_{\mathcal{D}_1}(\omega),\pi_{\mathcal{D}_2}(\omega)))\end{aligned}$$

Proposition 4 *Let $\mathcal{B}_1$, $\mathcal{B}_2$ be two possibilistic bases, and let $\pi_{\mathcal{B}_{S-C}}$ be the possibility distribution obtained by Equation (10) and $\pi_{\mathcal{B}_{dm}}$ be the possibility distribution obtained by Equation (3), then $\pi_{\mathcal{B}_{S-C}}$ is more specific than $\pi_{\mathcal{B}_{dm}}$, that is $\pi_{\mathcal{B}_{S-C}}(\omega) \leq \pi_{\mathcal{B}_{dm}}(\omega)$ for all $\omega \in \Omega$.*

Proof. *Suppose $\pi_{\mathcal{B}_{dm}}(\omega)\neq 1$, then by equation (1),*

$$\begin{aligned}\pi_{\mathcal{B}_1}(\omega) &= min_{i=1,n}\{1-\alpha_i|\omega \models \neg\phi_i, \phi_i \in \mathcal{B}_1\}\\ &= min(min_{i=1,n}\{1-\alpha_i|\omega \models \neg\phi_i,\ \phi_i \in \mathcal{C}_1\},\\ &\quad min_{i=1,n}\{1-\alpha_i| \models \neg\phi_i,\ \phi_i \in \mathcal{D}_1\})\\ &= min(\pi_{\mathcal{C}_1}(\omega),\pi_{\mathcal{D}_1}(\omega)).\end{aligned}$$

Similarly, $\pi_{\mathcal{B}_2}(\omega) = min(\pi_{\mathcal{C}_2}(\omega), \pi_{\mathcal{D}_2}(\omega))$. *Therefore,*

$$\begin{aligned}\pi_{\mathcal{B}_{dm}}(\omega) &= max(\pi_{\mathcal{B}_1}(\omega),\pi_{\mathcal{B}_2}(\omega))\\ &= max(min(\pi_{\mathcal{C}_1}(\omega),\ \pi_{\mathcal{D}_1}(\omega)),\\ &\quad min(\pi_{\mathcal{C}_2}(\omega),\ \pi_{\mathcal{D}_2}(\omega))).\end{aligned}$$

It is clear $\pi_{\mathcal{B}_{S-C}}(\omega) \leq \pi_{\mathcal{B}_{dm}}(\omega)$, so $\pi_{\mathcal{B}_{S-C}}$ is more specific than $\pi_{\mathcal{B}_{dm}}$.

T-Norm and T-conorm-Based S-C Combination Method

Some basic definitions

In (Benferhat, Dubois, and Prade 1997a), some triangular norm (T-norm for short) based combination modes are introduced to provide a reinforcement effect. Namely, if expert 1 assigns possibility $\pi_1(\omega) < 1$ to an interpretation ω, and expert 2 assigns possibility $\pi_2(\omega) < 1$ to this interpretation, then in some triangular norm modes, $\pi(\omega) < min(\pi_1(\omega), \pi_2(\omega))$, where π is the possibility distribution obtained by combining π_1 and π_2 using a triangular norm mode.

Definition 8 (Klement, Mesiar, and Paf 2000) *A triangular norm (T-norm) tn is a two place real-valued function $tn : [0,1] \times [0,1] \rightarrow [0,1]$ and which satisfies the following conditions:*

1. *tn(0,0)=0, and tn(a,1)=tn(1,a)=a, for every a (boundary condition);*
2. *tn(a,b)≤tn(c,d) whenever $a \leq c$ and $b \leq d$ (monotonicity);*
3. *tn(a,b)=tn(b,a) (symmetry);*
4. *tn(a,tn(b,c))=tn(tn(a,b),c) (associativity).*

A triangular conorm (T-conorm) ct is a two place real-valued function whose domain is the unit square [0,1]×[0,1] and which satisfies the conditions 2-4 given in the previous definition plus the following boundary conditions:

5. *ct(1,1)=1,ct(a,0)=ct(0,a)=a.*

Any T-conorm ct can be generated from a T-norm through the duality transformation:

$$ct(a,b) = 1 - tn(1-a, 1-b)$$

and conversely.

It is easy to check that the maximum operator is a T-conorm and the minimum operator is a T-norm. Other frequently used T-norms are the product operator and the *Lukasiewicz T-norm* ($max(0, a+b-1)$). The duality relation respectively yields the following T-conorm: the *probabilistic sum* ($a + b - ab$), and the *bounded sum* ($min(1, a + b)$).

Given two possibilistic bases $\mathcal{B}_1$ and $\mathcal{B}_2$ with possibility distributions $\pi_{\mathcal{B}_1}$ and $\pi_{\mathcal{B}_2}$ respectively, the semantic results of their combination by a T-norm tn and a T-conorm ct are

$$\forall\omega,\ \pi_{tn}(w) = tn(\pi_{\mathcal{B}_1}(w), \pi_{\mathcal{B}_2}(w)), \qquad (11)$$

$$\forall\omega,\ \pi_{ct}(w) = ct(\pi_{\mathcal{B}_1}(w), \pi_{\mathcal{B}_2}(w)). \qquad (12)$$

The syntactic results associated with π_{tn} and π_{ct} are respectively the following knowledge bases (Benferhat, Dubois, and Prade 1997a):

$$\begin{aligned}\mathcal{B}_{tn} &= \mathcal{B}_1 \cup \mathcal{B}_2 \cup \{(\phi_i \vee \psi_j,\ ct(\alpha_i,\beta_j))|(\phi_i,\ \alpha_i) \in \mathcal{B}_1\\ &\quad and\ (\psi_j,\ \beta_j) \in \mathcal{B}_2\}, \qquad (13)\end{aligned}$$

$$\begin{aligned}\mathcal{B}_{ct} &= \{(\phi_i \vee \psi_j,\ tn(\alpha_i,\ \beta_j))|(\phi_i,\ \alpha_i) \in \mathcal{B}_1\\ &\quad and\ (\psi_j,\ \beta_j) \in \mathcal{B}_2\}. \qquad (14)\end{aligned}$$

Syntactic and semantic results of the T-S-C combination method

In Step 2 of the S-C combination algorithm, if we replace the *maximum* operator by the *T-conorm* and replace the *minimum* operator by the *T-norm*, then we get a more general S-C combination mode, we call it *T-norm and T-conorm-based split-combination mode*(or T-S-C combination mode for short). By Equation (13) and Equation (14), the syntactic results of the combination of $\mathcal{C}_1$ and $\mathcal{C}_2$ by T-conorm and the combination of $\mathcal{D}_1$ and $\mathcal{D}_2$ by T-norm are

$$\begin{aligned}\mathcal{C} &= \{(\phi_i \vee \psi_j,\ tn(\alpha_i,\ \beta_j))|(\phi_i,\ \alpha_i) \in \mathcal{C}_1\ and\\ &\quad (\psi_j,\ \beta_j) \in \mathcal{C}_2\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{D} &= \mathcal{D}_1 \cup \mathcal{D}_2 \cup \{(\phi_i \vee \psi_j,\ ct(\alpha_i,\beta_j))|\\ &\quad (\phi_i,\ \alpha_i) \in \mathcal{D}_1\ and\ (\psi_j,\ \beta_j) \in \mathcal{D}_2\}\end{aligned}$$

respectively. Therefore, the syntactic result of combination of $\mathcal{B}_1$ and $\mathcal{B}_2$ by the T-S-C combination mode is

$$\begin{aligned}\mathcal{B}_{T-S-C} &= \{(\phi_i \vee \psi_j,\ tn(\alpha_i,\ \beta_j))|(\phi_i,\ \alpha_i) \in \mathcal{C}_1\\ &\quad and\ (\psi_j,\ \beta_j) \in \mathcal{C}_2\} \cup \mathcal{D}_1 \cup \mathcal{D}_2 \cup\\ &\quad \{(\phi_i \vee \psi_j,\ ct(\alpha_i,\beta_j))|(\phi_i,\ \alpha_i) \in \mathcal{D}_1\\ &\quad and\ (\psi_j,\ \beta_j) \in \mathcal{D}_2\} \qquad (15)\end{aligned}$$

Proposition 5 *Given two possibilistic bases $\mathcal{B}_1$ and $\mathcal{B}_2$, if $\mathcal{B}_{T-S-C}$ is the possibilistic base obtained by Equation (15) and $\mathcal{B}_{ct}$ is the possibilistic base obtained by Equation (14), then every formula in $\mathcal{B}_{ct}$ can be inferred from $\mathcal{B}_{T-S-C}$, namely*

$$\mathcal{B}_{T-S-C} \vdash_{\pi} (\phi,\ \alpha), \quad \textit{for all } (\phi,\ \alpha) \in \mathcal{B}_{ct} \tag{16}$$

Proof. *Each formula in $\mathcal{B}_{ct}$ has the form $(\phi_i \vee \psi_j,\ tn(\alpha_i, \beta_j))$, so we consider four cases:*

Case 1: *$\alpha_i, \beta_j \leq Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, then $(\phi_i,\ \alpha_i) \in \mathcal{C}_1$ and $(\psi_j,\ \beta_j) \in \mathcal{C}_2$. Therefore, $(\phi_i \vee \psi_j,\ tn(\alpha_i,\beta_j) \in \mathcal{B}_{T-S-C}$ and we have $\mathcal{B}_{T-S-C} \vdash_{\pi} (\phi_i \vee \psi_j,\ tn(\alpha_i,\beta_j))$.*

Case 2: *$\alpha_i > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$ and $\beta_j \leq Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, then $tn(\alpha_i,\beta_j) \leq min\{\alpha_i,\beta_j\} < \alpha_i$. Since $(\phi_i,\ \alpha) \in \mathcal{D}_1$ and $\phi_i \models \phi_i \vee \psi_j$, we have $\mathcal{B}_{T-S-C} \vdash_{\pi} (\phi_i \vee \psi_j,\ tn(\alpha_i,\beta_j))$.*

Case 3: *$\alpha_i \leq Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$ and $\beta_j > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, this case is a dual to case 2.*

Case 4: *$\alpha_i > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$ and $\beta_j > Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, then since $(\phi_i \vee \psi_j,\ ct(\alpha_i,\beta_j)) \in \mathcal{D}$ and $ct(\alpha_i,\beta_j) \geq tn(\alpha_i,\beta_j)$, we have $\mathcal{B}_{T-S-C} \vdash_{\pi} (\phi_i \vee \psi_j,\ tn(\alpha_i,\beta_j))$.*

This completes our proof.

Example 4 *(Continue Example 2) $\mathcal{B}_1$ and $\mathcal{B}_2$ are split as $< \mathcal{C}_1, \mathcal{D}_1 >$ and $< \mathcal{C}_2, \mathcal{D}_2 >$ such that*

$$\mathcal{C}_1 = \{(\phi,\ 0.2), (\psi,\ 0.4)\},\ \ \mathcal{D}_1 = \{(\gamma,\ 0.7)\}$$

and

$$\mathcal{C}_2 = \{(\neg\psi,\ 0.5), (\gamma,\ 0.3)\},\ \ \mathcal{D}_2 = \{(\phi,\ 0.6)\}.$$

In this example, we choose max(0,$a + b - 1$) which is the Lukasiewicz T-norm as T-norm and min(1,$a+b$) which is the bounded sum as T-conorm in the second fusion step. Then combining $\mathcal{C}_1$ and $\mathcal{C}_2$ by the bounded sum and combining $\mathcal{D}_1$ and $\mathcal{D}_2$ by the Lukasiewicz T-norm we obtain $\mathcal{C}$ and $\mathcal{D}$ such that

$$\mathcal{C} = \{(\phi \vee \gamma,\ 0), (\phi \vee \neg\psi,\ 0), (\psi \vee \gamma,\ 0)\} = \emptyset,$$
$$\mathcal{D} = \{(\gamma,\ 0.7), (\phi\ 0.6), (\psi \vee \phi,\ 1)\}.$$

So the syntactic result of combination of $\mathcal{B}_1$ and $\mathcal{B}_2$ is

$$\mathcal{B}_{T-S-C} = \{(\gamma,\ 0.7), (\phi,\ 0.6), (\psi \vee \phi,\ 1)\}.$$

By Equation (14), $\mathcal{B}_{ct}$ obtained by combining $\mathcal{B}_1$ and $\mathcal{B}_2$ by T-conorm operator min(1,$a + b$) is $\mathcal{B}_{ct} = \{(\phi,\ 0), (\phi \vee \neg\psi,\ 0), (\phi\vee\gamma,\ 0), (\psi\vee\phi,\ 0), (\psi\vee\gamma,\ 0), (\phi\vee\gamma,\ 0.3), (\neg\psi\vee \gamma,\ 0.2), (\gamma,\ 0)\} = \{(\phi \vee \gamma,\ 0.3), (\neg\psi \vee \gamma,\ 0.2)\}$. It is easy to check that all the formulas in $\mathcal{B}_{ct}$ can be inferred from $\mathcal{B}_{T-S-C}$.

Given two possibilistic bases $\mathcal{B}_1$ and $\mathcal{B}_2$, after Step 1 in the T-S-C combination algorithm, $\mathcal{B}_1$ and $\mathcal{B}_2$ are split into

$$\mathcal{B}_1 = \mathcal{C}_1 \cup \mathcal{D}_1\ and\ \mathcal{B}_2 = \mathcal{C}_2 \cup \mathcal{D}_2.$$

Suppose $\pi_{\mathcal{C}_1},\ \pi_{\mathcal{C}_2},\ \pi_{\mathcal{D}_1},\ \pi_{\mathcal{D}_2}$ are possibility distributions of $\mathcal{C}_1,\ \mathcal{C}_2,\ \mathcal{D}_1,\ \mathcal{D}_2$ respectively. Then by Equation (12) and Equation (11), the combination of $\mathcal{C}_1$ and $\mathcal{C}_2$ by the T-conorm is $\mathcal{C}$ such that $\forall\ \omega, \pi_{\mathcal{C}}(w) = ct(\pi_{\mathcal{C}_1}(w), \pi_{\mathcal{C}_2}(w))$ and the combination of $\mathcal{D}_1$ and $\mathcal{D}_2$ by the T-norm is $\mathcal{D}$ such that $\forall\ \omega, \pi_{\mathcal{D}}(w) = tn(\pi_{\mathcal{D}_1}(w), \pi_{\mathcal{D}_2}(w))$. Since $\mathcal{B}_{T-S-C} = \mathcal{C} \cup \mathcal{D}$, we have $\forall\ \omega, \pi_{\mathcal{B}_{T-S-C}}(w) = min\{\pi_{\mathcal{C}}(w), \pi_{\mathcal{D}}(w)\} = min\{ct(\pi_{\mathcal{C}_1}(w), \pi_{\mathcal{C}_2}(w)), tn(\pi_{\mathcal{D}_1}(w), \pi_{\mathcal{D}_2}(w))\}$.

An Alternative Way to Split Possibilistic Bases

An alternative splitting approach

In the S-C algorithm, given two possibilistic bases $\mathcal{B}_1$ and $\mathcal{B}_2$, we split each of them using the *upper free degree* of their union, such that $\mathcal{B}_1 = \mathcal{C}_1 \cup \mathcal{D}_1$ and $\mathcal{B}_2 = \mathcal{C}_2 \cup \mathcal{D}_2$. Then we combine $\mathcal{C}_1$ and $\mathcal{C}_2$ by the *maximum* operator (or more generally a *T-conorm* operator) to deal with inconsistency. Since $\mathcal{C}_1 \cup \mathcal{C}_2$ consists of possibilistic formulas in $\mathcal{B}_1 \cup \mathcal{B}_2$ with necessity degrees less than $Free_{upp}(\mathcal{B}_1 \cup \mathcal{B}_2)$, there may exist some *free formulas* in $\mathcal{C}_1$ or $\mathcal{C}_2$. So, when we combine $\mathcal{C}_1$ and $\mathcal{C}_2$ by the *maximum* operator, these *free formulas* are combined with other formulas as disjunctive forms. However, we know *free formulas* will not cause inconsistency, so it is safe to keep them unchanged. Therefore, we propose the following approach to split the knowledge bases.

Definition 9 *(free-formulas-based splitting) Given a possibilistic base $\mathcal{B}$, the splitting of $\mathcal{B}$ with regard to $Free(\mathcal{B})$ is a pair $< \mathcal{C}_{Con}, \mathcal{D}_{Free} >$ such that $\mathcal{B} = \mathcal{C}_{Con} \cup \mathcal{D}_{Free}$, where*

$$\mathcal{D}_{Free} = \{(\phi,\ \alpha) | (\phi,\ \alpha) \in Free(\mathcal{B})\},$$
$$\mathcal{C}_{Con} = \mathcal{B} \setminus \mathcal{D}_{Free} = \{(\phi,\ \alpha) | (\phi,\ \alpha) \notin Free(\mathcal{B})\}.$$

That is, $\mathcal{D}_{Free}$ contains all the free formulas, and $\mathcal{C}_{Con}$ contains all the conflict formulas in $\mathcal{B}$.

Lemma 2 *Let $\mathcal{B}$ be a possibilistic knowledge base. Let $\mathcal{B}$ be split by upper-free-degree approach and free-formulas approach respectively, with the splitting results as $\mathcal{B} = \mathcal{C} \cup \mathcal{D}$ and $\mathcal{B} = \mathcal{C}_{Con} \cup \mathcal{D}_{Free}$. Then $\mathcal{D} \subseteq \mathcal{D}_{Free}$, and $\mathcal{C}_{Con} \subseteq \mathcal{C}$.*

Proof. *Let $(\phi,\ \alpha) \in \mathcal{D}$, by Definition 5 and Definition 6, all the formulas in $\mathcal{D}$ are free in $\mathcal{B}$, so $(\phi,\ \alpha)$ is a free formula in $\mathcal{B}$. Therefore $(\phi,\ \alpha) \in \mathcal{D}$ and $\mathcal{D} \subseteq \mathcal{D}$. On the other hand, $\mathcal{C}_{Con} = (\mathcal{B} \setminus \mathcal{D}) \subseteq (\mathcal{B} \setminus \mathcal{D}) = \mathcal{C}$.*

Now we revise the split-combination algorithm by replacing the *upper free degree*-based splitting approach with the *free-formulas*-based splitting approach.

Free-formulas-based-split-combination(F-S-C Combination) Algorithm

- **Step 0:** Let $\mathcal{B}_1 = \{(\phi_1,\ \alpha_1), ..., (\phi_n,\ \alpha_n)\}$ and $\mathcal{B}_2 = \{(\psi_1,\ \beta_1), ..., (\psi_m,\ \beta_m)\}$ be two possibilistic bases, compute $Free(\mathcal{B}_1 \cup \mathcal{B}_2)$.
- **Step 1:**Split $\mathcal{B}_1$ and $\mathcal{B}_2$ with regard to $Free(\mathcal{B}_1 \cup \mathcal{B}_2)$ as follows. Let $< \mathcal{C}', \mathcal{D}' >$ be a splitting of $\mathcal{B}_1 \cup \mathcal{B}_2$ with regard to $Free(\mathcal{B}_1 \cup \mathcal{B}_2)$. The splitting of $\mathcal{B}_1$ is a pair $< \mathcal{C}_1, \mathcal{D}_1 >$ such that $\mathcal{C}_1 = \mathcal{C}' \cap \mathcal{B}_1$ and $\mathcal{D}_1 = \mathcal{D}' \cap \mathcal{B}_1$. The splitting of $\mathcal{B}_2$ is a pair $< \mathcal{C}_2, \mathcal{D}_2 >$ such that $\mathcal{C}_2 = \mathcal{C}' \cap \mathcal{B}_2$ and $\mathcal{D}_2 = \mathcal{D}' \cap \mathcal{B}_2$.
- **Step 2:** Combine $\mathcal{C}_{Con_1}$ and $\mathcal{C}_{Con_2}$ by the maximum operator, and combine $\mathcal{D}_{Free_1}$ and $\mathcal{D}_{Free_2}$ by the minimum operator. The results are $\mathcal{C}_{Con}$ and $\mathcal{D}_{Free}$ respectively. By Equation (4) and Equation (5) we have

$$\mathcal{C}_{Con} = \{(\phi_i \vee \psi_j,\ min(\alpha_i, \beta_j) | (\phi_i,\ \alpha_i) \in \mathcal{C}_{Con_1}, (\psi_j,\ \beta_j) \in \mathcal{C}_{Con_2}\}, \tag{17}$$

$$\mathcal{D}_{Free} = \mathcal{D}_{Free_1} \cup \mathcal{D}_{Free_2}. \tag{18}$$

- **Step 3** The final result of the F-S-C combination method, denoted by $\mathcal{B}_{F-S-C}$, is $\mathcal{C}_{Con} \cup \mathcal{D}_{Free}$.

Lemma 3 *The possibilistic base $\mathcal{B}_{F-S-C}$ obtained by Step 3 in the F-S-C combination algorithm is consistent.*

Proof. *The proof of lemma 3 is similar to that of Proposition 2.*

Proposition 6 *Given two possibilistic bases $\mathcal{B}_1$ and $\mathcal{B}_2$, if $\mathcal{B}_{F-S-C}$ is the possibilistic base obtained by the F-S-C combination mode and $\mathcal{B}_{S-C}$ is the possibilistic base obtained by the S-C combination mode, then*

$$\mathcal{B}_{F-S-C} \vdash_\pi (\phi,\ \alpha), \quad \text{for all } (\phi,\ \alpha) \in \mathcal{B}_{S-C}. \tag{19}$$

Proof. *Let $(\gamma,\ \delta) \in \mathcal{B}_{S-C}$, since $\mathcal{B}_{S-C} = \mathcal{C} \cup \mathcal{D}$, where $\mathcal{C}$ is obtained by Equation (7) and $\mathcal{D}$ is obtained by Equation (8), we have $(\gamma,\ \delta) \in \mathcal{C} \cup \mathcal{D}$. On the one hand, suppose $(\gamma,\ \delta) \in \mathcal{D}$, then $(\gamma,\ \delta) \in \mathcal{D}_1 \cup \mathcal{D}_2$. By Lemma 2, $\mathcal{D}_1 \subseteq \mathcal{D}_{Free_1}$ and $\mathcal{D}_2 \subseteq \mathcal{D}_{Free_2}$. So $(\gamma,\ \delta) \in \mathcal{D}_{Free_1} \cup \mathcal{D}_{Free_2} = \mathcal{D}_{Free}$. Since $\mathcal{B}_{F-S-C} = \mathcal{C}_{Con} \cup \mathcal{D}_{Free}$, we have $(\gamma,\ \delta) \in \mathcal{B}_{F-S-C}$, so $\mathcal{B}_{F-S-C} \vdash_\pi (\gamma,\ \delta)$. On the other hand, suppose $(\gamma,\ \delta) \in \mathcal{C}$, then $(\gamma,\ \delta)$ has the form $(\phi_i \vee \psi_j,\ min(\alpha_i, \beta_j))$, where $(\phi_i,\ \alpha_i) \in \mathcal{C}_1$ and $(\psi_j,\ \beta_j) \in \mathcal{C}_2$. By Lemma 2, $\mathcal{C}_{Con_1} \subseteq \mathcal{C}_1$ and $\mathcal{C}_{Con_2} \subseteq \mathcal{C}_2$. We consider following two cases:*

Case 1: *$(\phi_i,\ \alpha_i) \in \mathcal{C}_{Con_1}$ and $(\psi_j,\ \delta_j) \in \mathcal{C}_{Con_2}$. In this case, we have $(\gamma,\ \delta) = (\phi_i \vee \psi_j,\ min(\alpha_i, \beta_j)) \in \mathcal{C}_{Con}$. So $(\gamma,\ \delta) \in \mathcal{B}_{F-S-C}$ and $\mathcal{B}_{F-S-C} \vdash_\pi (\gamma,\ \delta)$.*

Case 2: *$(\phi_i,\ \alpha_i) \notin \mathcal{C}_{Con_1}$ or $(\psi_j,\ \beta_j) \notin \mathcal{C}_{Con_2}$. Assume $(\psi_j,\ \beta_j) \notin \mathcal{C}_{Con_2}$(for $(\phi_i,\ \alpha_i) \notin \mathcal{C}_{Con_1}$, the proof is similar). In this case, $(\psi_j,\ \beta_j) \in \mathcal{D}_{Free}$, so $(\psi_j,\ \beta_j) \in \mathcal{B}_{F-S-C}$. Since $\psi_j \models \gamma$, and $\beta_j \geq min(\alpha_i, \beta_j) = \delta$, we have $\mathcal{B}_{F-S-C} \vdash_\pi (\gamma,\ \delta)$.*

This completes our proof.

Example 5 *(Continue Example 3) $Free(\mathcal{B}_1 \cup \mathcal{B}_2) = \{(\neg\phi \vee \gamma,\ 0.8), (\phi \vee \delta,\ 0.9), (\neg\psi \vee \delta,\ 0.7), (\delta \vee \gamma,\ 0.5)\}$. So $\mathcal{B}_1$ and $\mathcal{B}_2$ are split as $< \mathcal{C}_{Con_1}, \mathcal{D}_{Free_1} >$ and $< \mathcal{C}_{Con_2}, \mathcal{D}_{Free_2} >$, where*

$$\mathcal{C}_{Con_1} = \{(\phi,\ 0.5), (\neg\phi \vee \psi,\ 0.6)\},$$
$$\mathcal{D}_{Free_1} = \{(\neg\phi \vee \gamma,\ 0.8)\}.$$

and

$$\mathcal{C}_{Con_2} = \{(\psi,\ 0.3), (\neg\phi \vee \neg\psi,\ 0.4)\},$$
$$\mathcal{D}_{Free_2} = \{(\phi \vee \delta,\ 0.9), (\neg\psi \vee \delta,\ 0.7), (\delta \vee \gamma,\ 0.5)\}.$$

By Equation (17) and Equation(18) we have

$$\mathcal{C}_{Con} = \{(\phi \vee \psi,\ 0.3), (\neg\phi \vee \psi,\ 0.3)\},$$

$$\mathcal{D}_{Free} = \{(\neg\phi \vee \gamma,\ 0.8), (\phi \vee \delta,\ 0.9), (\neg\psi \vee \delta,\ 0.7), (\delta \vee \gamma,\ 0.5)\}.$$

Therefore

$$\mathcal{B}_{F-S-C} = \{(\phi \vee \psi,\ 0.3), (\neg\phi \vee \psi,\ 0.3), (\neg\phi \vee \gamma,\ 0.8), (\phi \vee \gamma,\ 0.9), (\neg\psi \vee \delta,\ 0.7), (\delta \vee \gamma,\ 0.5)\},$$

which is equivalent to $\{(\neg\phi \vee \gamma,\ 0.8), (\psi,\ 0.3), (\phi \vee \delta,\ 0.9), (\neg\psi \vee \delta,\ 0.7)\}$. It is easy to check that $\mathcal{B}_{F-S-C} \vdash_\pi (\phi,\ \alpha)$ for any $(\phi,\ \alpha) \in \mathcal{B}_{S-C}$.

Application of F-S-C combination method to merge flat knowledge bases

It has been pointed out in (Dubois, Lang, and Prade 1994) that when the necessity degrees of all the possibilistic formulas are taken as 1, possibilistic logic will regress to classical logic. So classical logic is a special case of possibilistic logic in which all the formulas have the same level of priority. That is, given a set of formulas $F = \{\phi_1, ..., \phi_n\}$ in classical logic, we can relate it with a set of possibilistic formulas $\mathcal{F} = \{(\phi_1,\ 1), ...(\phi_n,\ 1)\}$. Therefore, our F-S-C combination method can be applied to merge flat (or classical) knowledge bases.

In (Benferhat, Dubois, and Prade 1997b), a consequence relation called *free consequence relation* is defined to cope with inconsistency in flat knowledge bases.

Definition 10 *A formula ϕ is said to be a free consequence of a flat knowledge base $\mathcal{B}$, denoted $\mathcal{B} \models_{Free} \phi$, if and only if ϕ is logically entailed from $Free(\mathcal{B})$, namely,*

$$\mathcal{B} \models_{Free} \phi, \quad iff \quad Free(\mathcal{B}) \models \phi$$

Given two flat knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$, a method was introduced in (Benferhat, Dubois, and Prade 1997b) which concatenated $\mathcal{B}_1 \cup \mathcal{B}_2$, i.e., the result of merging is $\mathcal{B}_1 \cup \mathcal{B}_2$. When $\mathcal{B}_1 \cup \mathcal{B}_2$ was inconsistent, some inconsistency tolerant consequence relations, for example, the free consequence relation, could be used to deal with it.

Proposition 7 *Given two flat knowledge bases $\mathcal{B}_1$ and $\mathcal{B}_2$, every free consequence of $\mathcal{B}_1 \cup \mathcal{B}_2$ can be inferred from $\mathcal{B}_{F-S-C}$.*

Proof. *When applying the F-S-C combination algorithm to merge $\mathcal{B}_1$ and $\mathcal{B}_2$ by taking all the necessity degrees of the formulas in $\mathcal{B}_1$ and $\mathcal{B}_2$ as 1, we obtain*

$$\mathcal{B}_{F-S-C} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \{\phi \vee \psi | \phi \in \mathcal{B}_1, \psi \in \mathcal{B}_2, \phi, \psi \notin Free(\mathcal{B}_1 \cup \mathcal{B}_2)\}.$$

Since $\mathcal{D}_1 \cup \mathcal{D}_2 = Free(\mathcal{B}_1 \cup \mathcal{B}_2)$, we have

$$\mathcal{B}_{F-S-C} = Free(\mathcal{B}_1 \cup \mathcal{B}_2) \cup \{\phi \vee \psi | \phi \in \mathcal{B}_1, \psi \in \mathcal{B}_2, \phi, \psi \notin Free(\mathcal{B}_1 \cup \mathcal{B}_2)\}.$$

So $Free(\mathcal{B}_1 \cup \mathcal{B}_2) \subseteq \mathcal{B}_{F-S-C}$. If γ is a $free\ consequence$ of $\mathcal{B}_1 \cup \mathcal{B}_1$, then $Free(\mathcal{B}_1 \cup \mathcal{B}_2) \vdash \gamma$. Therefore, $\mathcal{B}_{F-S-C} \vdash \gamma$.

The proof of proposition 7 shows that $\mathcal{B}_{F-S-C}$ keeps all the free formula unchanged, and combine all the subbases containing conflict formulas. By contrast, if we combine B_1 and B_2 by concatenation and deal with the inconsistency using the free consequence relation, then only free formulas are used and the conflict formulas are ignored. Consequently, the converse of proposition 7 is false.

Example 6 *Given two flat bases $\mathcal{B}_1 = \{\phi,\ \neg\phi \vee \neg\psi\}$, $\mathcal{B}_2 = \{\psi,\ \neg\phi \vee \delta,\ \psi \vee \delta\}$, the free base of $\mathcal{B}_1 \cup \mathcal{B}_2$ is $Free(\mathcal{B}_1 \cup \mathcal{B}_2) = \{\neg\phi \vee \delta,\ \psi \vee \delta\}$. Splitting $\mathcal{B}_1$ and $\mathcal{B}_2$ with regard to $Free(\mathcal{B}_1 \cup \mathcal{B}_2)$, we have $\mathcal{B}_1 = \mathcal{C}_{Con_1} \cup \mathcal{D}_{Free_1}$ such that $\mathcal{C}_{Con_1} = \{\phi,\ \neg\phi \vee \neg\psi\}$ and $\mathcal{D}_{Free_1} = \emptyset$, and $\mathcal{B}_2 = \mathcal{C}_{Con_2} \cup \mathcal{D}_{Free_2}$ such that $\mathcal{C}_{Con_2} = \{\psi\}$ and $\mathcal{D}_{Free_2} = \{\neg\phi \vee \delta,\ \psi \vee \delta\}$. Then we combine $\mathcal{C}_{Con_1}$ and $\mathcal{C}_{Con_2}$ by* *the maximum combination mode and combine $\mathcal{D}_{Free_1}$ and $\mathcal{D}_{Free_2}$ by the minimum combination mode, the results are*

$$\mathcal{C}_{Con} = \{\phi \vee \psi\} \; and \; \mathcal{D}_{Free} = \{\neg\phi \vee \delta, \ \neg\psi \vee \delta\}.$$

So the possibilistic base of combination of $\mathcal{B}_1$ and $\mathcal{B}_2$ by the F-S-C combination mode is $\mathcal{B}_{F-S-C} = \{\phi\vee\psi, \ \neg\phi\vee\delta, \ \psi\vee\delta\}$.

In (Baral et al. 1992), some methods to combine knowledge bases consisting of first order theories are introduced. Given *n* classical knowledge bases, one method is to take the union of them and select all the maximal consistent subbases [1] from the union to resolve the inconsistency, i.e., a formula is a maximal-consistent-subbase based consequence (MCS-consequence for short) of the union iff it can be inferred from every maximal consistent subbases of the union. The following example shows that our method is not comparable with the MCS-consequence based method.

Example 7 *Given two classical knowledge bases $\mathcal{B}_1 = \{\phi, \neg\phi \vee \neg\psi, \gamma\}$ and $\mathcal{B}_2 = \{\psi, , \neg\phi \vee \neg\gamma\}$. Since $\{\phi, \neg\phi \vee \psi, \psi\}$ and $\{\phi, \neg\phi \vee \neg\gamma, \gamma\}$ are two minimal inconsistent subbases of $\mathcal{B}_1 \cup \mathcal{B}_2$, $Free(\mathcal{B}_1 \cup \mathcal{B}_2) = \emptyset$. By the F-S-C algorithm, $\mathcal{B}_{F-S-C} = \{\phi \vee \psi, \neg\phi \vee \neg\psi \vee \neg\gamma, \psi \vee \gamma\}$. By contrast, $\mathcal{B}_1 \cup \mathcal{B}_2$ contains five maximal consistent knowledge bases $B_1 = \{\phi, \psi, \gamma\}$, $B_2 = \{\phi, \psi, \neg\phi \vee \neg\gamma\}$, $B_3 = \{\phi, \gamma, \neg\phi \vee \neg\psi\}$, $B_4 = \{\phi, \neg\phi \vee \neg\psi, \neg\phi \vee \neg\gamma\}$, $B_5 = \{\psi, \gamma, \neg\phi \vee \neg\psi, \neg\phi \vee \neg\gamma\}$. It is easy to check that $\psi \vee \gamma$ can not be inferred from B_4, therefore, it is not a MCS-consequence of $\mathcal{B}_1 \cup \mathcal{B}_2$. However, $\psi \vee \gamma \in \mathcal{B}_{F-S-C}$, so it can be inferred from $\mathcal{B}_{F-S-C}$. Conversely, $\phi \vee \gamma$ can be inferred from each B_i, so it is a MCS-consequence of $\mathcal{B}_1 \cup \mathcal{B}_2$. However, $\phi \vee \gamma$ can not be inferred from $\mathcal{B}_{F-S-C}$.*

Conclusions

In this paper we first proposed a new method for merging inconsistent possibilistic bases. Fusion of possibilistic bases is completed by two steps. In the first step, each of the possibilistic bases is split into two subbases with regard to the upper free degree of their union, such that one subbase contains formulas whose necessity degrees are less than the *upper free degree* and the other contains formulas whose necessity degrees are greater than the *upper free degree*. Then in the second step, we combine those subbases containing formulas with necessity degree less than the *upper free degree* using the maximum (or more generally, *T-conorm*) combination mode, while combining the subbases containing formulas with necessity degree greater than the *upper free degree* using the minimum (or more generally, *T-norm*) combination mode. The union of the possibilistic bases obtained by the second step is taken as the result of the combination of the possibilistic bases we want to merge. We proved that the combination mode obtained by the new method is better than the maximum combination mode. We then proposed another splitting method, called a *Free-formulas-based* splitting. The combination algorithm, called a F-S-C combination algorithm, obtained by the *Free-formulas-based* splitting can be applied to merge knowledge bases which are flat. We proved that the F-S-C method is better than the *free consequence* based merging method in (Benferhat, Dubois, and Prade 1997b).

[1] A maximal consistent subbases X of a knowledge base $\mathcal{B}$ is a consistent subbase such that none of the consistent subbase of $\mathcal{B}$ contains X.

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