Rough operations on Boolean algebras

Guilin Qi * and Weiru Liu

School of Computer Science, Queen's University Belfast
Belfast, BT7 1NN, UK

Abstract

In this paper, we introduce two pairs of rough operations on Boolean algebras. First we define a pair of rough approximations based on a partition of the *unity* of a Boolean algebra. We then propose a pair of generalized rough approximations on Boolean algebras after defining a basic assignment function between two different Boolean algebras. Finally, some discussions on the relationship between rough operations and some uncertainty measures are given to provide a better understanding of both rough operations and uncertainty measures on Boolean algebras.

Key words: Boolean algebra; Rough operations; Uncertainty measures

1 Introduction

Rough set theory was introduced by Pawlak [16] to generalize the classical set theory. In rough set theory, given an equivalence relation on a universe, we can define a pair of rough approximations which provide a lower bound and an upper bound for each subset of the universe. Rough approximations can also be defined equivalently by a partition of the universe which is corresponding to the equivalence relation [11]. In [14,15], Yao generalized rough set theory by generalizing the equivalence relation to a more general relation, and then interpreted belief functions using the generalized rough set theory in [12].

In [6,7], Bayesian theory and Dempster-Shafer theory [3] are extended to be constructed on Boolean algebras. This provides a more general framework to deal with uncertainty reasoning. In this paper, we define a pair of dual rough

* Corresponding author. Tel.: +44-28-90974847; fax:+44-28-90975666
Email addresses: G.Qi@qub.ac.uk (Guilin Qi), W.Liu@qub.ac.uk (and Weiru Liu).

set operations on Boolean algebras to interpret belief functions and some other uncertainty measures on Boolean algebras. The new pair of rough operations coincide with $\cap$- and $\cup$-homomorphisms in *interval structure* [9,10]. The difference is that the $\cap$- and $\cup$-homomorphisms are defined by some axioms, whilst the (generalized) rough operations are defined by a (generalized) partition. The lower approximation defined in this paper can be used to interpret belief functions on Boolean algebras in [7]. We establish a one to one correspondence between belief functions and pairs consisting of a Bayesian function and a lower approximation given by a basic assignment. This result is more general than the corresponding relation between rough sets and belief functions given in [12].

This paper is organized as follows. In the first two sections, we introduce some important concepts of Boolean algebra and evidential measures on Boolean algebras. In Section 4, we first define a pair of rough approximations based on a partition of the unity of a Boolean algebra, then propose a pair of generalized rough approximations after defining a basic assignment function between two different Boolean algebras. In Section 5, we discuss relationship between rough operations and some uncertainty measures. Finally, we summary the paper in Section 6.

2 A Brief Review of Boolean Algebra

A Boolean algebra [4] is a 6-tuple $\langle \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi \rangle$ where χ is a set: $\Phi, \Psi \in \chi$, and for every $A,\ B \in \chi$ there exist $A \cup B \in \chi,\ A \cap B \in \chi$ satisfying the following conditions

b1) commutative laws: $A \cup B = B \cup A,\ A \cap B = B \cap A$;
b2) associative laws: $A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap B) \cap C$;
b3) there exist $\Phi,\ \Psi \in \chi:\ A \cup \Phi = A,\ A \cap \Psi = A$ and $\Phi \neq \Psi$;
b4) for every $A \in \chi$ there exists an $A' \in \chi$ such that $A \cup A' = \Psi,\ A \cap A' = \Phi$;
b5) distributive laws:
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),\ A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

We call

1. χ the *space* of the Boolean algebra $\langle \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi \rangle$;
2. $\cup$ the *union* operation of the algebra;
3. $\cap$ the *intersection* operation of the algebra;
4. $'$ the *negation* operation of the algebra;
5. Φ the *zero* element of the algebra;
6. Ψ the *unity* element of the algebra

We say that B *includes* A, denoted by $A \subseteq B$, if any one of the following statements hold: 1) $A \cap B = A$; 2) $A \cup B = B$; 3) $A' \cup B = \Psi$; 4) $A \cap B' = \Phi$.

If $\subseteq$ is considered as a partial ordering, we have

1) Φ is the smallest element in χ; i.e., Φ is included in every $A \in \chi$.
2) Ψ is the biggest element in χ; i.e., Ψ includes every $A \in \chi$.

Example 1 (Finite sets) Let Θ be a finite non-empty set, denoting the *frame of discernment*. Then $\langle 2^{\Theta},\ \cup,\ \cap,\ \Theta - X,\ \emptyset,\ \Theta \rangle$ is a Boolean algebra, where

1. 2^{Θ} is the *power set* of Θ; i.e., the set of subsets of Θ, $2^{\Theta} = \{X | X \subseteq \Theta\}$;
2. $\cup$ is the union operation for subsets, $X \cup Y = \{x \in \Theta | x \in X \textit{ or } x \in Y\}$;
3. $\cap$ is the intersection operation for subsets, $X \cap Y = \{x \in \Theta | x \in X \textit{ and } x \in Y\}$;
4. $\Theta - X$ is the complement of X, $\Theta - X = \{x \in \Theta | x \notin X\}$;
5. $\emptyset$ is the empty set, the least element in 2^{Θ} under the inclusive relation $\subseteq$;
6. Θ is the greatest element in 2^{Θ} under the corresponding inclusive relation $\subseteq$.

Example 2 (Propositions) Let $\mathcal{P}$ be a set of propositions. Then $\langle \mathcal{P},\ \vee,\ \wedge,\ \neg,\ F,\ T \rangle$ is a Boolean algebra, where F is the ever-false proposition, T is the ever-true proposition, $\vee$ is the disjunction operation, $\wedge$ is the conjunction operation, and $\neg$ is the negation operation.

In the case $\langle \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi \rangle = \langle \mathcal{P},\ \vee,\ \wedge,\ \neg,\ F,\ T \rangle$, we usually replace $A \subseteq B$ with $A \rightarrow B$.

A non-empty subset χ_0 of a Boolean algebra χ is said to be a *sub-algebra* of χ provided χ_0 is closed under the operations $\cup$, $\cap$, $-$, i. e. the following conditions are satisfied:

1) if $A,\ B \in \chi_0$, then $A \cup B \in \chi_0$;
2) if $A,\ B \in \chi_0$, then $A \cap B \in \chi_0$;
3) if $A \in \chi_0$, then $A' \in \chi_0$.

Each subalgebra χ_0 of any Boolean algebra χ is also a Boolean algebra under the same operations $\cup$, $\cap$, $'$ restricted to χ_0. The inclusion relation in the Boolean algebra χ_0 is that of χ, restricted to χ_0.

An element $a \neq \Phi$ of a Boolean algebra χ is said to be an *atom* of χ, if for every $A \in \chi$, the inclusion

$$A \subseteq a$$

implies that

$$\textit{either} \quad A = \Phi \quad \textit{or} \quad A = a.$$

The notion of atom is a Boolean analogue of one-point sets. If a is an atom of a Boolean algebra χ, then, for every element $B \in \chi$,

$$either \ \ a \subseteq B \ \ or \ \ a \cap B \ = \ \Phi.$$

3 Evidential Functions on Boolean Algebras

In this section, we introduce some uncertainty measures on Boolean algebras in [6,7].

Let $\langle \chi, \ \cup, \ \cap, \ ', \ \Phi, \ \Psi \rangle$ be a Boolean algebra, where Φ is the *zero element* of χ and Ψ is the *unity element* of χ.

A function $bay : \chi \rightarrow [0,1]$ is called a *Bayesian function* if

y1) $bay(\Phi) \ = \ 0,$
y2) $bay(\Psi) \ = \ 1.$
y3) $bay(A \cup B) = bay(A) + bay(B) - bay(A \cap B).$

A function $bew : \chi \rightarrow [0,1]$ is called a *weak belief function* if it satisfies

b1) $bew(\Phi) \ = \ 0,$
b2) $bew(\Psi) \ = \ 1,$
b3) for any collection $A_1, \ A_2, \ ..., A_n (n \geq 1)$ of subsets of Ψ.

$$bew(\bigcup_{i=1,...,n} A_i) \geq \sum_{I \subseteq \{1,...,n\}, \ I \neq \Phi} (-1)^{|I|+1} bew(\bigcap_{i \in I} A_i).$$

Given a weak belief function bew, the function $dow(A) = bew(A')$ is called a *weak doubt function* and the function $plw(A) = 1 - dow(A) = 1 - bew(A')$ is called a *weak plausibility function*.

A function $m : \chi \rightarrow [0,1]$ is called a *mass function* if it has non-zero value only at a finite number of elements $A_1, \ A_2, \ ..., \ A_F$ in χ and it satisfies

m1) $m(\Phi) = 0,$
m2) $\Sigma_{X \subseteq \Psi} m(X) \ = \ \Sigma_{i=1}^{F} m(A_i) = 1.$

A mass function is also called a *basic probability assignment*.

A function bel on χ is called a *belief function* if it can be expressed in terms of a mass function $m : bel(A) = \Sigma_{X \subseteq A} m(X)$ for all $A \in \chi$.

Theorem 3 [7] (Belief function is a weak belief function) Let m be a mass function. Then the function *bel* defined by the following expression

$$bel(A) = \Sigma_{X\subseteq A} m(X) \qquad \text{for all } A \subseteq \Psi$$

is a weak belief function.

4 Rough Operations on Boolean Algebras

In Pawlak's rough set theory, a rough set is induced by a partition of the universe. In this section, we will extend Pawlak's rough set theory by defining a pair of rough operations induced by a partition of the unity of a Boolean algebra. A partition in a Boolean algebra is defined as

Definition 4 (Partition of Ψ) Given a Boolean algebra $\langle \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi\rangle$, a family consisting of pairwise disjoint elements $\{A_i\}_{i\in I}$ (where I is an index set) is called a partition of Ψ if it satisfies

$$\bigcup_{i\in I} A_i = \Psi.$$

Now we can define a pair of rough operations on a Boolean algebra.

Definition 5 Let $\langle \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi\rangle$ be a Boolean algebra. If $\mathcal{A} = \{A_i\}_{i=1,\ldots,n}$ $(n\geq 1)$ is a partition of Ψ, then a pair of operations $L : \chi \to \chi$ and $H : \chi \to \chi$ such that

$$L(A) = \bigcup_{A_i\subseteq A,\ A_i\in\mathcal{A}} A_i, \tag{1}$$

$$H(A) = \bigcup_{A\cap A_i\neq\Phi,\ A_i\in\mathcal{A}} A_i \tag{2}$$

are called a lower approximation and an upper approximation respectively.

We give some of the properties of the lower approximation and upper approximation as follows.

Theorem 6 Let $\langle \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi\rangle$ be a Boolean algebra. If $\mathcal{A} = \{A_i\}_{i=1,\ldots,n}$ $(n\geq 1)$ is a partition of Ψ, L and H are the lower and upper approximations induced by $\mathcal{A}$ respectively, then we have:

1) $L(\Psi) = \Psi$
2) $L(A)\subseteq A$
3) $L(A) = L(L(A))$

4) $L(A \cap B) = L(A) \cap L(B)$
5) $L(A) = H(L(A))$
6) $L(A) = (H(A'))'$.

Proof. 1)-3) and 5) are clear by the definitions of partition and lower approximation. Now let us prove 4) and 6).

4)

$$\begin{aligned} L(A \cap B) &= \cup_{A_i \subseteq A \cap B} A_i \\ &= \cup_{A_i \subseteq A,\ and\ A_i \subseteq B} A_i \\ &= (\cup_{A_i \subseteq A} A_i) \cap (\cup_{A_i \subseteq B} A_i) \\ &= L(A) \cap L(B). \end{aligned}$$

6) Before the proof of 6), let us introduce a lemma.

Lemma 7 [4] Let $\langle \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi \rangle$ be a Boolean algebra, $A,\ B \in \chi$, then

$$A \cap B' = \Phi \ \ if\ and\ only\ if \ \ A \subseteq B.$$

Now we can continue the proof of 6)

$$\begin{aligned} (H(A'))' &= (\cup_{A_i \cap A' \neq \Phi} A_i)' \\ &= \cup_{A_i \cap A' = \Phi} A_i \\ &= \cup_{A_i \subseteq A} A_i \\ &= L(A). \end{aligned}$$

Example 8 For the Boolean algebra $\langle 2^{\Theta},\ \cup,\ \cap,\ \Theta - X,\ \emptyset,\ \Theta \rangle$ in Example 1, it is clear that, given a partition on Θ, the induced lower approximation and upper approximation are rough lower approximation and rough upper approximation in Pawlak's rough set algebra respectively.

Given two different Boolean algebras $\langle \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi \rangle$ and $\langle \mathcal{U},\ \cup,\ \cap,\ ',\ \perp,\ \top \rangle$, a mapping $j : \chi \to \mathcal{U}$ is called a basic assignment if it satisfies [10]:

1) $j(\Phi) = \perp$
2) $j(A) \cap j(B) = \perp, \ \ if \ \ A \neq B$
3) $\bigcup_{A \in \chi} j(A) = \top$.

If $j(A) \neq \perp$, A is called a focal element of j. Clearly, all the focal elements form a partition of $\top$. We call this partition a *generalized partition*. In the following, we always assume that the focal elements of j are finite.

Definition 9 Let $\langle \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi \rangle$ and $\langle \mathcal{U},\ \cup,\ \cap,\ ',\ \perp,\ \top \rangle$ be two Boolean algebras. Suppose $j : \chi \rightarrow \mathcal{U}$ is a basic assignment. Then a pair of operations $L : \chi \rightarrow \mathcal{U}$ and $H : \chi \rightarrow \mathcal{U}$ such that

$$L(A) = \bigcup_{B \subseteq A} j(B), \quad \text{for each } A \in \chi \tag{3}$$

$$H(A) = \bigcup_{B \cap A \neq \Phi} j(B), \quad \text{for each } A \in \chi \tag{4}$$

are called a generalized lower approximation and a generalized upper approximation respectively.

In Definition 9, if $\chi = \mathcal{U}$, $\{A_i\}_{i=1,\ldots,n}$ $(n \geq 1)$ is a partition of Ψ and we define $j(A) = A$ for each A in $\{A_i\}_{i=1,\ldots,n}$ $(n \geq 1)$, then the generalized lower approximation(generalized upper approximation) is lower approximation(upper approximation). Therefore, a generalized lower approximation(generalized upper approximation) is an extension of a lower approximation (upper approximation).

Given two Boolean algebras, if L and H are generalized lower and upper approximations defined by a basic assignment, then the interval $[L, H]$ is an interval structure defined in [9,10]. That is, the generalized lower and upper approximations coincide with $\cap$- and $\cup$-homomorphisms in interval structure. The difference is that the $\cap$- and $\cup$-homomorphisms are defined by some axioms, but the (generalized) approximations are defined by a (generalized) partition.

The following properties of generalized lower and upper approximations can be found in [10,13].

(1) $L(A) \cup L(B) \subseteq L(A \cup B)$,
(2) $L(A) \cap L(B) = L(A \cap B)$,
(3) $L(\Phi) = \perp$,
(4) $L(\Psi) = \top$,

and

(1′) $H(A \cap B) \subseteq H(A) \cap H(B)$,
(2′) $H(A) \cup H(B) = H(A \cup B)$,
(3′) $H(\Phi) = \perp$,

$(4') \quad H(\Psi) = \top,$

Generalized lower approximation and upper approximation can be defined axiomatically.

Definition 10 Suppose $L : \chi \to \mathcal{U}$ and $H : \chi \to \mathcal{U}$ are two mappings from a Boolean algebra $\langle \chi, \cup, \cap, ', \Phi, \Psi \rangle$ to another Boolean algebra $\langle \mathcal{U}, \cup, \cap, ', \bot, \top \rangle$. We say that L and H are dual mappings if $H(A) = (L(A'))'$ for every $A \in \chi$.

Definition 11 Let $\langle \chi, \cup, \cap, ', \Phi, \Psi \rangle$ and $\langle \mathcal{U}, \cup, \cap, ', \bot, \top \rangle$ be two Boolean algebras. A pair of dual mappings L, H are called generalized lower approximation and upper approximation respectively, if they satisfy the following properties:

L1) $L(\Psi) = \top$,
L2) $L(A \cap B) = L(A) \cap L(B)$,
H1) $H(\Phi) = \bot$,
H2) $H(A \cup B) = H(A) \cup H(B)$.

In fact, axioms L1) and L2) form an independent set of axioms for L, whereas H1) and H2) form an independent set of axioms for H.

There is a one to one corresponding relationship between basic assignment and generalized lower approximation.

Theorem 12 Let L and H be two mappings from a Boolean algebra χ to another Boolean algebra $\mathcal{U}$ with $L(A) = (H(A'))'$ for every $A \in \chi$. L satisfies axioms L1) and L2) if and only if there exists a basic assignment $j : \chi \to \mathcal{U}$ such that for all $A \in \chi$,

$$L(A) = \bigcup_{B \subseteq A} j(B), \tag{5}$$

and j is defined as

$$j(A) = L(A) \cap (\bigcup_{B \subseteq A} L(B))'. \tag{6}$$

The proof of Theorem 12 is similar to that of Theorem 1 in [10].

Example 13 (Generalized rough sets) Suppose $(2^U, \cup, \cap, \neg, \emptyset, U)$ and $(2^W, \cup, \cap, \neg, \emptyset, W)$ are two Boolean algebras, where U and W are two finite sets, and $j : 2^U \to 2^W$ is a basic assignment from 2^U to 2^W, then it is clear that the generalized lower approximation and upper approximation induced by j are the pair of lower and upper approximations in a rough set algebra $(2^W,\ 2^U,\ \cap,\ \cup,\ \neg,\ L,\ H)$([11]).

Example 14 (Incidence calculus) Suppose $\langle \mathcal{P},\ \wedge,\ \vee,\ \neg,\ F, T\rangle$ and $(2^\Theta,\ \cup,\ \cap,\ \Theta - X,\ \emptyset,\ \Theta)$ are Boolean algebras of propositions (see Example 2) and finite sets (see Example 1) respectively. If $j : \mathcal{P} \to 2^\Theta$ is a basic assignment, then the generalized lower approximation and upper approximation defined by

$$\begin{aligned} i_*(\phi) &= \bigcup_{\models \psi \to \phi} j(\psi) \\ i^*(\phi) &= \bigcup_{\psi \wedge \phi \neq F} j(\psi) \end{aligned}$$

are called lower and upper bounds on the incidence respectively. The tightest pair of lower and upper mappings ([1,2,10]) in incidence calculus are lower and upper bounds on the incidence defined above.

Example 15 (Modal algebra) In [8], a structure called modal algebra was introduced to give an algebraic semantics for modal logic [5]. A structure $\mathcal{M} = \langle M,\ \cup,\ \cap,\ -,\ P\rangle$ is a normal modal algebra iff M is a set of elements closed under operations $\cup,\ \cap,\ -$, and P such that:

(i) M is a Boolean algebra with respect to $\cup,\ \cap,\ -$;
(ii) for $x,\ y \in M$, $P(x \cup y) = Px \cup Py$.
(iii) $P(\Phi) = \Phi$, where Φ is zero element of M

Another dual operation N is defined as

$$Nx = -P - x.$$

Clearly, these defined operations P and N are a lower and an upper rough approximations respectively. Therefore, our newly defined rough approximations can be used to interpret modal logic.

5 Rough Operations and Uncertainty Measures

In this section, we will discuss the relationship between rough operations and uncertainty measures. In the first subsection, we introduce inner and outer measures on Boolean algebras. Then in the second subsection, the relationship between rough operations and uncertainty measures is examined.

Given a Boolean algebra, we may only know Bayesian functions on a subalgebra of it when information is absent. For those elements not belonging to the subalgebra, what we can do is to define inner measure and outer measure on them as follows:

Definition 16 Let $\langle \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi\rangle$ be a Boolean algebra, χ_0 be a subalgebra of it. If $bay : \chi_0 \rightarrow [0,1]$ is a Bayesian function on χ_0, a pair of functions $bay_* : \chi_0 \rightarrow [0,1]$ and $bay^* : \chi_0 \rightarrow [0,1]$ defined by

$$bay_*(A) = sup\{bay(U) | U \subseteq A, U \in \chi_0\} \tag{7}$$
$$bay^*(A) = inf\{bay(U) | A \subseteq U, U \in \chi_0\} \tag{8}$$

are called inner and outer measures induced by a Bayesian function *bay* respectively.

The inner and outer measures of a subset A of Ψ can be viewed as our best estimate of the *true* measures of A, given our lack of knowledge. Suppose χ is a Boolean algebra, and χ' is a subalgebra of it. We say that Bayesian functions *bay* on χ and bay' on χ' agree on χ' if $bay(A) = bay'(A)$ for all $A \in \chi'$. Clearly we have the following result: If Bayesian functions *bay* on χ and bay' on χ' agree on χ', then $bay'_*(A) \leq bay(A) \leq bay'^*(A)$.

Now let us discuss some important properties of inner and outer measures.

Theorem 17 Let $< \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi >$ be a Boolean algebra, and χ_0 be a subalgebra of it. If *bay* is a Bayesian function on χ_0, bay_* and bay^* are inner and outer measures induced by *bay* respectively, then the followings hold:

(1) if $A \subseteq B$, then $bay_*(A) \leq bay_*(B)$, $bay^*(A) \leq bay^*(B)$.
(2) $bay_*(A) \leq bay^*(A)$.
(3) $bay_*(A) = 1 - bay^*(A')$.
(4) $bay_*(A \cup B) \geq bay_*(A) + bay_*(B) - bay_*(A \cap B)$.

Proof. (1) and (2) are clear. We will prove (3) and (4).

(3) By Definition 16, we have

$$\begin{aligned} 1 - bay^*(A') &= 1 - inf\{bay(U) | A' \subseteq U,\ U \in \chi_0\} \\ &= sup\{1 - bay(U) | U' \subseteq A,\ U \in \chi_0\} \\ &= sup\{bay(U') | U' \subseteq A,\ U \in \chi_0\} \\ &= bay_*(A). \end{aligned}$$

(4)

$$
\begin{aligned}
&bay_*(A{\cup}B) = sup\{bay(U)|U{\subseteq}A{\cup}B,\ U \in \chi_0\}\\
&{\geq}sup\{bay(C{\cup}D)|C{\subseteq}A,\ D{\subseteq}B,\ where\ C,\ D \in \chi_0\}\\
&= sup\{bay(C) + bay(D) - bay(C{\cap}D)|C{\subseteq}A,\ D{\subseteq}B,\ C,\ D \in \chi_0\}\\
&{\geq}sup\{bay(C) + bay(D) - bay_*(A{\cap}B)|C{\subseteq}A,\ D{\subseteq}B,\ C,\ D \in \chi_0\}\\
&= sup\{bay(C)|C{\subseteq}A,\ C \in \chi_0\} + sup\{bay(D)|D{\subseteq}B,\ D \in \chi_0\}\\
&\quad -bay_*(A{\cap}B)\\
&= bay_*(A) + bay_*(B) - bay_*(A{\cap}B).
\end{aligned}
$$

Next we will discuss the relationship between inner measures and belief functions.

Theorem 18 Let $\langle\chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi\rangle$ be a Boolean algebra, and χ_0 be a subalgebra of it. If χ_0 is finite, then every inner measure bay_* on χ induced by a Bayesian function bay on χ_0 is a belief function.

Proof. Since χ_0 is finite, there exists a subset $\mathcal{Y}$ of χ_0 such that for every $A{\in}\chi_0$, there exist some elements $\{B_i\}_{i=1,\dots m}$ in $\mathcal{Y}$ such that $A = \cup_{i=1}^m B_i$ and $\mathcal{Y}$ is a partition of Ψ (we can take $\mathcal{Y}$ as all the atoms of χ_0). If we define $m(A) = bay(A)$ for $A{\in}\mathcal{Y}$ and $m(A) = 0$ otherwise, then it is easy to check that m is a mass function on χ. Moreover,

$$
\begin{aligned}
bay_*(A) &= max\{bay(B)|B{\subseteq}A,\ B{\in}\chi_0\}\\
&= \sum_{B\subseteq A,\ B\in\mathcal{Y}} bay(B)\\
&= \sum_{B\subseteq A,\ B\in\mathcal{Y}} m(B)
\end{aligned}
$$

Therefore, bay_* is a belief function.

Since a belief function must be a weak belief function, we have the following corollary.

Corollary 19 *Let $\langle\chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi\rangle$ be a Boolean algebra, and χ_0 be a subalgebra of it. If χ_0 is finite, then every inner measure bay_* on χ induced by a Bayesian function bay on χ_0 is a weak belief function.*

Let $\langle \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi\rangle$ be a Boolean algebra , and $\mathcal{Y}$ be a finite partition of Ψ. Let us define $\chi_0 = \{A|A$ is the union of some elements in $\mathcal{Y}\}$, then it is easy to check that χ_0 is a subalgebra of χ. Moreover, suppose $\mathcal{U}$ is an arbitrary subalgebra of χ containing $\mathcal{Y}$, then we have $\chi_0 \subseteq \mathcal{U}$. Therefore, χ_0 defined above is the smallest subalgebra of χ containing $\mathcal{Y}$. If *bay* is a Bayesian function defined on χ_0, then we can extend *bay* by defining

$$bay_*(A) = bay(L(A)), \tag{9}$$
$$bay^*(A) = bay(H(A)), \tag{10}$$

where A is a subset of Ψ, $L(A)$ and $H(A)$ are lower and upper approximations of A respectively. Moreover, we have the following theorem.

Theorem 20 bay_* and bay^* defined by Equation (9) and (10) are inner and outer measures respectively.

Proof. Since

$$\begin{aligned} bay_*(A) &= bay(L(A)) \\ &= bay(\cup_{A_i \subseteq A,\ A_i \in \mathcal{Y}} A_i) \\ &= max\{bay(U)|U \subseteq A,\ U \in \chi_0\}, \end{aligned}$$

and

$$\begin{aligned} bay^*(A) &= bay(H(A)) \\ &= bay(\cup_{A_i \cap A \neq \emptyset,\ A_i \in \mathcal{Y}} A_i) \\ &= min\{bay(U)|A \subseteq U,\ U \in \chi_0\}, \end{aligned}$$

it follows that bay_* and bay^* are inner and outer measures respectively.

By Theorem 18, Corollary 19 and Theorem 20 we have following conclusion.

Corollary 21 bay_* defined by Equation (9) is both a weak belief function and a belief function.

From the proof of Theorem 18, clearly we have the following theorem.

Theorem 22 Let $\langle \chi,\ \cup,\ \cap,\ ',\ \Phi,\ \Psi\rangle$ be a Boolean algebra, and χ_0 be a subalgebra of it. If χ_0 is finite, then for each inner measure bay_* induced by some Bayesian function bay, there exists a partition $\mathcal{Y}$ of Ψ such that

$$bay_*(A) = bay(L(A)), \quad \text{for every } A \in \chi,$$

where $L(A)$ is the lower approximation induced by $\mathcal{Y}$.

Theorem 20 and Theorem 22 tell us that the lower approximation and the inner measure have a one to one corresponding relationship.

When two Boolean algebras are given, we get a pair of generalized rough operations derived from a basic assignment. We have the following theorem.

Theorem 23 Let $\langle \chi, \cup, \cap, ', \Phi, \Psi \rangle$ and $\langle \mathcal{U}, \cup, \cap, ', \perp, \top \rangle$ be two Boolean algebras. Suppose $j : \chi \to \mathcal{U}$ is a basic assignment, and L is the generalized lower approximation derived from j. Suppose bay is a Bayesian function on $\mathcal{U}$, then the function bel defined by

$$bel(A) \;=\; bay(L(A)), for\ every\ A \in \chi \tag{11}$$

is a belief function.

Proof. We define a function m as

$$m(A) = bay(j(A)), for\ every\ A \in \chi.$$

It is easy to check m defined above is a basic probability assignment. Let M denote the set containing all the focal elements of j. Then

$$\begin{aligned} bel(A) &= bay(L(A)) \\ &= \sum_{B \subseteq A,\ B \in M} bay(j(B)) \\ &= \sum_{B \subseteq A,\ B \in M} m(B), \end{aligned}$$

so bel is a belief function.

Conversely, given a belief function on a Boolean algebra, we can define a Bayesian function and a generalized lower approximation such that Equation (11) holds.

Theorem 24 Let $\langle \chi, \cup, \cap, ', \Phi, \Psi \rangle$ be a Boolean algebra. If bel is a belief function defined on χ, then there exists a Boolean algebra $\mathcal{U}$ and a Bayesian function bay on it such that Equation (11) holds.

Proof. Suppose m is the corresponding mass function of bel, M is the set of focal elements of m and $M = \{A_1, A_2, ...A_n\}$. Let $\mathcal{U}$ be a Boolean algebra such that there are at least n atoms in it (this Boolean algebra must exist, because we can take U as a finite set with at least n elements, then define $\mathcal{U} = 2^U$). Let $\mathcal{A} = \{U_1, ..., U_n\}$ be a partition of Ψ derived from atoms (this can be done by uniting some atoms). If $\mathcal{U}_0$ is a subalgebra generated by $\mathcal{A}$ (that is, $\mathcal{U}_0$ is the smallest subalgebra of $\mathcal{U}$ containing $\mathcal{A}$), then we can define a mapping $j : \chi \rightarrow \mathcal{U}_0$ such that $j(A) = U_i$ for $A \in M$ and $j(A) = \emptyset$ otherwise. Clearly j is a basic assignment on χ to $\mathcal{U}_0$. If we define a Bayesian function bay on $\mathcal{U}_0$ such that

$$bay(U_i) = m(A_i),$$

then it is clear

$$bel(A) = bay(L(A)), \quad \text{for each } A \in \chi.$$

This completes the proof.

Theorem 23 and Theorem 24 show that belief functions are in one to one correspondence with pairs consisting of a Bayesian function and a lower approximation given by a basic assignment.

6 Summary

In this paper, we introduced a pair of dual rough operations on Boolean algebras and used them to interpret some uncertainty measures on Boolean algebras. This pair of rough operations (called generalized lower and upper approximations) coincide with $\cap$-homomorphism and $\cup$-homomorphism defined in *interval structure* [9,10]. We established a one to one correspondence relation between belief functions and pairs of a Bayesian function and a lower approximation. This correspondence relation is more general than the one established in [12], where both rough sets and belief functions are defined on classical sets.

References

[1] A. Bundy, Correctness Criteria of some algorithms for uncertain reasoning, Journal of Automated Reasoning 2 (1986) 109-126.

[2] A. Bundy, Incidence Calculus, in: The Encyclopedia of AI 1992, pp. 663-668.

[3] G. Shafer, A Mathematical Theory of Evidence, Princeton University Press, Princeton, NJ, 1976.

[4] Roman Sikorski, Boolean Algebra, Springer-Verlag, Berlin, 1960.

[5] B.F. Chellas, Modal Logic: An introduction, Cambridge University Press, Cambridge, 1980.

[6] J.W. Guan, D.A. Bell, Bayesian probability on Boolean algebra and applications to decision theory, Information Sciences 97 (1997) 267-292.

[7] J.W. Guan, D.A. BELL, The Dempster-Shafer theory on Boolean algebras, Chinese Journal of Advanced Software Research 3 (1996) 313-343.

[8] E.J. Lemmon, Algebraic sematics for modal logic I, Journal of Symoblic Logic 31 (1966) 46-65.

[9] S.K.M. Wong, L.S. Wang, and Y.Y. Yao, Interval structure: a framework for representing uncertain information, in: Proceedings of the 8th Conference on Uncertainty in Artificial Intelligence, Stanford University, California, USA, 1992, 336-343.

[10] S.K.M. Wong, L.S. Wang, and Y.Y. Yao, On modeling uncertainty with interval structures, Computational Intelligence 11 (1995) 406-426.

[11] Y.Y. Yao, Two views of the theory of rough sets in finite universes, International Journal of Approximation Reasoning, 15 (1996) 291-317.

[12] Y.Y. Yao and P.J. Lingras, Interpretations of belief functions in the theory of rough sets, Information Sciences 104(1998) 81-106.

[13] Y.Y. Yao, Constructive and algebraic methods of the theory of rough sets, Information Sciences 109 (1998) 21-47.

[14] Y.Y. Yao, Generalized rough set models, in: Rough Sets in Knowledge Discovery, Polkowski, L. and Skowron, A. (Eds.), Physica-Verlag, Heidelberg, 1998, pp. 286-318, .

[15] Y.Y. Yao, On generalizing Pawlak approximation operators, in: Polkowski, L. and Skowron, A. (Eds.), Proceedings of the First International Conference on Rough Sets and Current Trends in Computing (RSCTC'98), 1998, pp. 298-307.

[16] Z. Pawlak, Rough sets, International Journal of Computer and Information Science 11 (1981) 341–356.