Conflict Analysis and Merging Operators Selection in Possibility Theory

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Abstract. In possibility theory, *the degree of inconsistency* is commonly used to measure the level of conflict in information from multiple sources after merging, especially conjunctive merging. However, as shown in [HL05,Liu06b], this measure alone is not enough when pairs of uncertain information have the same degree of inconsistency, since it is not possible to tell which pair contains information that is actually *better*, in the sense that the two pieces of information in one pair agree with each other more than the information does in other pairs. In this paper, we investigate what additional measures can be used to judge the *closeness* between two pieces of uncertain information. We deploy the concept of *distance between betting commitments* developed in DS theory in [Liu06a], since possibility theory can be viewed as a special case of DS theory. We present properties that reveal the interconnections and differences between the degree of inconsistency and the distance between betting commitments. We also discuss how to use these two measures together to guide the possible selection of various merging operators in possibility theory.

1 Introduction

Pieces of uncertain information that come from different sources often do not agree with each other completely. There can be many reasons for this, such as, inaccuracy in sensor data reading, natural errors occurring in experiments, and unreliability of sources. When inconsistent information needs to be merged, assessing the degree of conflict among information plays a crucial role when deciding which combination mode would suit the best [DP94].

In possibility theory, the two basic combination modes are *conjunctive* and *disjunctive*, each of which has some specific merging operators. Some conjunctive operators also have reinforcement effects and they are more suitable to combine information that is highly consistent. In general, conjunctive operators are advised to combine information that is reliable and consistent and disjunctive operators are advised to merge inconsistent information [BDKP02]. The degree of inconsistency of merged information is widely used to judge how consistent that two (or multiple) pieces of possibilistic information are. Clearly, this value is not sufficient when multiple pairs of uncertain information have the same degree of inconsistency. We need additional approaches to measuring the degree of agreement (or conflict) between two pieces of possibilistic information in order to accurately decide which merging operator is more suitable, especially when an reinforcement operator is to be used.

In this paper, we take the advantage that possibility theory can be regarded as a special case of Dempster-Shafer theory and investigate how the degree of agreement (or conflict) between possibilistic uncertain information can be assessed by the concept of *distance between betting commitments* proposed in [Liu06a]. We particularly study the relationships between these two measures and are able to provide the following findings.

First, when a pair of possibilistic uncertain information is totally contradictory with each other, both measures give the same result, i.e., the maximum value of conflict. Second, when a pair of possibilistic uncertain information appears to be consistent, i.e., the degree of inconsistency is zero, the range of values of the distance between betting commitments can vary from zero to almost one. This finding is important since it tells that these two measures reveal two different aspects of the information involved. Third, when the degree of inconsistency is sufficiently large, the distance between betting commitments increases proportionally, that is the latter is a function of the former. Based on these findings, we are able to provide a set of more detailed guidelines as which conjunctive (or reinforcement) merging operator is more suitable for combining a given pair of uncertain information.

We will proceed as follows: in Section 2, we review the basics in possibility theory and DS theory and their connections. In Section 3, we investigate the relationships between the degree of inconsistency and the distance between betting commitments. In Section 4, we first review the general guidelines about how to select a merging operator in possibility theory (or possibilistic logic), we then provide a set of refined guidelines for this purpose. Finally in Section 5, we summarize the main contributions of the paper.

2 Preliminaries

2.1 Possibility theory

Possibility theory (or possibilistic logic) is a popular choice for representing uncertain information (or knowledge) ([DP82,BDP97], etc). At the semantic level, a basic function in possibility theory is a **possibility distribution** denoted as π which assigns each possible world in set Ω a value in $[0, 1]$ (or a set of graded values).

From a possibility distribution, a possibility measure (denoted as Π) and a necessity measure (denoted as N) can be derived as

$$\Pi(A) = \mathsf{max}(\{\pi(\omega) | \omega \in A\}) \text{ and } N(A) = 1 - \Pi(\bar{A}), \bar{A} = \Omega \setminus A \tag{1}$$

The former estimates to what extent the true event is believed to be in the subset and the latter evaluates the degree of necessity that the subset is true.

For a given π, if there exists $\omega_0 \in \Omega$ such that $\pi(\omega_0) = 1$, then π is said to be normal, otherwise, π is not normal. The value $1 - \mathsf{max}_{\omega \in \Omega} \pi(\omega)$ is called **the degree of inconsistency** of the information (or possibility distribution).

In possibility theory, the two families of merging operators are conjunctive and disjunctive. Examples of conjunctive operators are min, product and linear product and an example of disjunctive operator is max. Given two possibility distributions

π_1 and π_2, the semantic results of applying these operators are $\forall\omega \in \Omega$, $\pi_{\mathsf{min}}(\omega) = \mathsf{min}(\pi_1(\omega), \pi_2(\omega))$, $\pi_\times(\omega) = \pi_1(\omega) \times \pi_2(\omega)$, $\pi_\otimes(\omega) = \mathsf{max}(0, \pi_1(\omega) + \pi_2(\omega) - 1)$, and $\pi_{\mathsf{max}}(\omega) = \mathsf{max}(\pi_1(\omega), \pi_2(\omega))$, where we use $\times$ and $\otimes$ for product and linear product operators respectively.

2.2 Basics of Dempster-Shafer theory

In the Dempster-Shafer theory of evidence (DS theory) [Sha76], a piece of uncertain information is represented by a **basic probability assignment** (or called a **mass function**) m on a set (Ω) containing mutually exclusive and exhaustive solutions to a question. Ω is called the **frame of discernment**.

A mass function $m : 2^\Omega \to [0, 1]$ satisfies $m(\emptyset) = 0$ and $\sum_{A\subseteq\Omega} m(A) = 1$ (though condition $m(\emptyset) = 0$ is not strictly required in the Transferable Belief Model (TBM) [SK94]).

From m, a **belief function**, $Bel(A) : 2^\Omega \to [0, 1]$ is defined as $Bel(A) = \Sigma_{B\subseteq A} m(B)$. When $m(A) > 0$, A is referred to as a *focal element* of the belief function (by abuse of language, we simply say A is a focal element of mass function m in the rest of the paper). A **plausibility function** $Pl : 2^\Omega \to [0, 1]$ from m is defined as $Pl(A) = \Sigma_{B\cap A\neq\emptyset} m(B)$.

Two mass functions from distinct sources are usually combined using Dempster's combination rule. The rule is stated as follows.

Definition 1. *Let m_1 and m_2 be mass functions, and let $m_1 \oplus m_2$ be the combined mass function.*

$$m_1 \oplus m_2(C) = \frac{\Sigma_{A\cap B=C}\ (m_1(A) \times m_2(B))}{1 - \Sigma_{A\cap B=\emptyset}\ (m_1(A) \times m_2(B))}$$

when $\Sigma_{A\cap B=\emptyset}\ (m_1(A) \times m_2(B)) \neq 1$.

$\sum_{B\cap C=\emptyset} m_1(B)m_2(C)$ is the mass of the combined belief assigned to the emptyset before normalization and we denote it as $m_\oplus(\emptyset)$. In the following, whenever we use $m_\oplus(\emptyset)$, we always associate it with this explanation unless otherwise explicitly stated.

Definition 2. *[Sme04] Let m be a mass function on Ω. Its associated* **pignistic probability function** $\mathsf{BetP}_m : \Omega \to [0, 1]$ *is defined as*

$$\mathsf{BetP}_m(\omega) = \sum_{A\subseteq\Omega,\omega\in A} \frac{m(A)}{|A|}$$

where $|A|$ is the cardinality of subset A.

The transformation from m to BetP_m is called the **pignistic transformation**. In the original definition [Sme04], when $m(\emptyset) \neq 0$, $m(A)$ is replaced by $\frac{m(A)}{1-m(\emptyset)}$ in the above definition. Furthermore, $\mathsf{BetP}_m(A) = \sum_{\omega\in A} \mathsf{BetP}_m(\omega)$ for $A \subseteq \Omega$.

Definition 3. *([Liu06a]) Let m_1 and m_2 be two mass functions on frame Ω and let BetP_{m_1} and BetP_{m_2} be their corresponding pignistic probability functions respectively. Then*

$$\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} = \mathsf{max}_{A\subseteq\Omega}(|\mathsf{BetP}_{m_1}(A) - \mathsf{BetP}_{m_2}(A)|)$$

is called the **distance between betting commitments** *of the two mass functions.*

Value ($|\mathsf{BetP}_{m_1}(A) - \mathsf{BetP}_{m_2}(A)|$) is *the difference between betting commitments to A* from the two sources. The distance of betting commitments is therefore the maximum extent of the differences between betting commitments to all the subsets. $\mathsf{difBetP}^{\mathsf{m_2}}_{\mathsf{m_1}}$ is simplified as $\mathsf{difBetP}$ when there is no confusion as which two mass functions are being compared.

2.3 DS theory versus possibility theory

It has long been recognized that possibility theory is a special case of DS theory in the sense that from a possibility distribution, a mass function with nested focal elements can be recovered from it (e.g., [DP82]). In this case, a belief function is a necessity measure and a plausibility function is a possibility measure. The actual procedure to recover a mass function (and hence a belief function) is stated in the following definition.

Definition 4. *([DP82,DP88]) Let π be a possibility distribution on frame of discernment Ω and be normal. Let the set of values $\pi(\omega_i)$ be $\{\alpha_i | i = 1, ..., p\}$ and they are arranged as $\alpha_1 = 1 \geq \alpha_2 \geq \alpha_3, ..., \geq \alpha_p > 0$ and $\alpha_{p+1} = 0$. Let*

1. *$A_i = \{\omega | \pi(\omega) \geq \alpha_i\}$ for $i = 1, 2, ..., p$, then subsets $A_1, A_2, .., A_p$ are nested;*
2. *$m(A_i) = \pi(\omega_i) - \pi(\omega_{i+1})$ for $i = 1, 2, ..., p$, where $\omega_i \in A_i, \omega_{i+1} \in A_{i+1}$.*

Then m is a mass function recovered from π with focal elements A_i ($i = 1..., p$).

Example 1. Let π be a possibility distribution on $\Omega = \{\omega_1, ..., \omega_4\}$ where

$$\pi(\omega_1) = 0.7, \pi_2(\omega_2) = 1.0, \pi_2(\omega_3) = 0.8, \pi_2(\omega_4) = 0.7$$

Then the focal elements are $A_1 = \{\omega_2\}$, $A_2 = \{\omega_2, \omega_3\}$, and $A_3 = \Omega$. The corresponding mass function is $m(A_1) = 0.2$, $m(A_2) = 0.1$, and $m(A_3) = 0.7$.

3 Relationship between $\mathsf{Inc}(\pi)$, $\mathsf{difBetP}$ and $m_\oplus(\emptyset)$

Since $\mathsf{Inc}(\pi)$, $\mathsf{difBetP}$ and $m_\oplus(\emptyset)$ are developed for measuring inconsistency/conflict in possibility theory and DS theory respectively, and these two theories have some interconnections, we study formally the relationships among these three values.

Proposition 1. *([Liu06b]) Let π be a possibility distribution on frame of discernment Ω and be normal. Let BetP_m be the pignistic probability function of the corresponding mass function m derived from π. Then $\mathsf{BetP}_m(\omega_i) \geq \mathsf{BetP}_m(\omega_j)$ iff $\pi(\omega_i) \geq \pi(\omega_j)$.*

This proposition says that the more plausible a possible world is, the more betting commitment it carries. It is consistent with the *ordinal faithfulness* [Dub06] where a probability distribution preserves the ordering of possibilities of elementary events[1].

[1] It should be noted that in [Dub06], *ordinal faithfulness* refers to the preservation of the ordering of elementary events after transforming a probability distribution to a possibility distribution. Since obtaining BetP_m from a π satisfies this feature, we think it is worth to mention it here.

Example 2. (Con't Example 1) *Following Example 1, the pignistic probability function for the given possibility distribution is*

$\mathsf{BetP}_m(\omega_1) = 0.7/4$; $\mathsf{BetP}_m(\omega_2) = 0.2 + 0.1/2 + 0.7/4$;
$\mathsf{BetP}_m(\omega_3) = 0.1/2 + 0.7/4$; $\mathsf{BetP}_m(\omega_4) = 0.7/4$.

That is $\mathsf{BetP}_m(\omega_2) > \mathsf{BetP}_m(\omega_3) > \mathsf{BetP}_m(\omega_1) = \mathsf{BetP}_m(\omega_4)$ *which is consistent with the ordering of* $\pi(\omega_2) > \pi(\omega_3) > \pi(\omega_1) = \pi(\omega_4)$.

Proposition 2. *Let π_1 and π_2 be two possibility distributions on frame of discernment Ω and be normal. Let π_{min}, $\pi_\times$ and $\pi_\otimes$ be their merged results using the* min*, the* product*, and the* linear product *operators respectively. Then the following properties hold*

$$\mathsf{Inc}(\pi_{\mathsf{min}}) = 1 \textit{ iff } \mathsf{Inc}(\pi_\times) = 1 \textit{ iff } \mathsf{Inc}(\pi_\otimes) = 1$$

$$\mathsf{Inc}(\pi_{\mathsf{min}}) = 0 \textit{ iff } \mathsf{Inc}(\pi_\times) = 0 \textit{ iff } \mathsf{Inc}(\pi_\otimes) = 0$$

The proof of this proposition is straightforward and it enables us to prove the following propositions by using the min as the representative of conjunctive operators.

Proposition 3. *([Liu06b]) Let π_1 and π_2 be two possibility distributions on Ω and be normal. Let $\pi_\wedge$ be their conjunctively merged possibility distribution. Assume m_1 and m_2 are the mass functions derived from π_1 and π_2 respectively. Then the following properties hold*

1. $\mathsf{Inc}(\pi_\wedge) = 0$ *iff* $m_\oplus(\emptyset) = 0$
2. $\mathsf{Inc}(\pi_\wedge) = 1$ *iff* $m_\oplus(\emptyset) = 1$
3. $\mathsf{Inc}(\pi_\wedge) > 0$ *iff* $m_\oplus(\emptyset) > 0$

If we have two pairs of possibility distributions and we use $\pi^1_\wedge$ and $\pi^2_\wedge$ to denote their conjunctively merged possibility distributions, then $\mathsf{Inc}(\pi^1_\wedge) \geq \mathsf{Inc}(\pi^2_\wedge)$ does not imply $m^1_\oplus(\emptyset) \geq m^2_\oplus(\emptyset)$ in general, where $m^1_\oplus$ and $m^2_\oplus$ are the combined mass functions from the two pairs of mass functions derived from corresponding possibility distributions. This is demonstrated by Example 4 (in Section 4) where the two sets of possibility distributions have the same degree of inconsistency (0.2) but with different values assigned to the emptyset after combination (0.07 versus 0.23).

Proposition 4. *Let π_1 and π_2 be two possibility distributions and normal and let their conjunctively combined possibility distribution be $\pi_\wedge$. Furthermore, let m_1 and m_2 be their corresponding mass functions. Then we have the following property*

$$\mathsf{Inc}(\pi_\wedge) = 1 \textit{ iff } \mathsf{difBetP}^{\mathsf{m_2}}_{\mathsf{m_1}} = 1$$

Proof We take $\pi_\wedge = \pi_{\mathsf{min}}$ below without losing generality (see Proposition 2).

We first prove that $\mathsf{Inc}(\pi) = 1$ implies $\mathsf{difBetP}^{\mathsf{m_2}}_{\mathsf{m_1}} = 1$.

Let π_1 and π_2 be two possibility distributions, where π_{min} is the conjunctively merged distribution using min. When $\mathsf{Inc}(\pi_{\mathsf{min}}) = 1$, π_1 and π_2 are totally inconsistent, then for any $\omega \in \Omega$ either $\pi_1(\omega) = 0$ or $\pi_2(\omega) = 0$ or both. Let A_p and A_q be the largest focal elements of m_1 and m_2 respectively, then $A_p \cap A_q = \emptyset$, and both $\mathsf{BetP}_{m_1}(A_p) = 1$ and $\mathsf{BetP}_{m_2}(A_q) = 1$ hold. So we have $\mathsf{BetP}_{m_1}(A_p) - \mathsf{BetP}_{m_2}(A_p) = 1$, since we must have $\mathsf{BetP}_{m_2}(A_p) = 0$ when $\mathsf{BetP}_{m_2}(A_q) = 1$ (remember BetP is a probability function). Therefore $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} = 1$ must be true.

Next, we prove that $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} = 1$ implying $\mathsf{Inc}(\pi) = 1$. When $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} = 1$, there exists a subset $A \subset \Omega$ such that $\mathsf{BetP}_{m_2}(A) = 1$ and $\mathsf{BetP}_{m_1}(A) = 0$ (or vise versa). This means that A is the largest focal element for m_2 which implies $\forall \omega \in A$, $\pi_2(\omega) \neq 0$ and $\forall \omega \notin A, \pi_2(\omega) = 0$. On the other hand, $\mathsf{BetP}_{m_1}(A) = 0$ tells us that $\forall \omega \in A$, $\pi_1(\omega) = 0$. Therefore, we have

$\forall \omega \in A, \pi_{\mathsf{min}}(\omega) = \mathsf{min}(\pi_1(\omega), \pi_2(\omega)) = 0,$ since $\pi_1(\omega) = 0$, and

$\forall \omega \notin A, \pi_{\mathsf{min}}(\omega) = \mathsf{min}(\pi_1(\omega), \pi_2(\omega)) = 0,$ since $\pi_2(\omega) = 0$.

That is $\forall \omega \in \Omega, \pi_{\mathsf{min}}(\omega) = 0$. So $\mathsf{Inc}(\pi_{\mathsf{min}}) = 1$, and so is $\mathsf{Inc}(\pi_\wedge) = 1$.

$\diamond$

Propositions 3 and 4 together tell us that if two pieces of information contradict with each other completely, any of the three measures (i.e., $\mathsf{Inc}(\pi)$, $m_\oplus(\emptyset)$, or $\mathsf{difBetP}_{m_1}^{m_2}$) is sufficient to quantitatively justify it.

In general, $\mathsf{Inc}(\pi_\wedge) = 0 \Rightarrow \mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} = 0$ does not hold as shown below.

Example 3. Let two possibility distributions be

$$\pi_1(\omega_1) = 1.0, \pi_1(\omega_2) = 0.1, \pi_1(\omega_3) = 1.0, \pi_1(\omega_4) = 0.8;$$

$$\pi_2(\omega_1) = 1.0, \pi_2(\omega_2) = 0.9, \pi_2(\omega_3) = 0.2, \pi_2(\omega_4) = 0.1.$$

Then the degree of inconsistency between them is 0 *if they are merged conjunctively. Their corresponding mass functions are*

$$m_1(\{\omega_1, \omega_3\}) = 0.2, m_1(\{\omega_1, \omega_3, \omega_4\}) = 0.7, m_1(\{\Omega\}) = 0.1;$$

$$m_2(\{\omega_1\}) = 0.1, m_2(\{\omega_1, \omega_2\}) = 0.7, m_2(\{\omega_1, \omega_2, \omega_3\}) = 0.1, m_2(\{\Omega\}) = 0.1.$$

As we can see at least $\mathsf{BetP}_{m_2}(\omega_2) - \mathsf{BetP}_{m_1}(\omega_2) > 0$*, so* $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} = 0$ *does not hold.*

Proposition 5. *Let* π_1 *and* π_2 *be two possibility distributions on* Ω *and normal, and let their conjunctively combined possibility distribution be* $\pi_\wedge$*. Furthermore, let* m_1 *and* m_2 *be their corresponding mass functions. When* $\mathsf{Inc}(\pi_\wedge) = 0$ *we have*

$$0 \leq \mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} \leq \frac{(n-1)}{n}, \textit{where } n = |\Omega| \,.$$

Proof When two possibility distributions π_1 and π_2 are identical, $\mathsf{Inc}(\pi_\wedge) = 0$ must be true. Also, they generate the same mass function, and the same pignistic probability function, so $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} = 0$ holds in this situation. We have shown that $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} > 0$ is possible when $\mathsf{Inc}(\pi_\wedge) = 0$ in Example 3, therefore, $0 \leq \mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}}$ is true for any pair of possibility distributions when $\mathsf{Inc}(\pi_\wedge) = 0$.

Now, we prove that $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} \leq \frac{(n-1)}{n}$.

$\mathsf{Inc}(\pi_\wedge) = 0$ implies that there is at least one $\omega \in \Omega$ such that $\pi_1(\omega) = 1$ and $\pi_2(\omega) = 1$.

First, we consider a situation where there is only one element in Ω, denoted as ω_1 such that $\pi_1(\omega_1) = 1$ and $\pi_2(\omega_1) = 1$. We further assume that for all $\omega_i \in \Omega$, $\pi_1(\omega_i) = 0$ if $\omega_i \neq \omega_1$ and $\pi_2(\omega_i) = 1$. Then the two mass functions from these two possibility distributions are $m_1(\{\omega_1\}) = 1$ and $m_2(\Omega) = 1$. Therefore

$$\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} = \frac{(n-1)}{n}$$

because $\mathsf{BetP}_{m_1}(\Omega \setminus \{\omega_1\}) = 0$ and $\mathsf{BetP}_{m_2}(\Omega \setminus \{\omega_1\}) = \frac{(n-1)}{n}$.

Before proving that for any two possibility distributions that $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} \leq \frac{(n-1)}{n}$ holds, we need to prove that for a positive integer $n > 2$, the following inequality is true

$$\frac{n-1}{n} > \frac{n-2}{n-1}$$

This is obvious since $(n-1)^2 > n(n-2)$. Therefore, we have

$$\frac{n-1}{n} > \frac{n-2}{n-1} > \frac{n-3}{n-2} > ... > \frac{1}{2} \tag{2}$$

Next, we proof that for any π_1 and π_2 with their $\mathsf{Inc}(\pi_\wedge) = 0$, $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} \leq \frac{(n-1)}{n}$.

For this case, we still assume that $\pi_1(\omega_1) = 1$ and $\pi_2(\omega_1) = 1$, because we have the assumption $\mathsf{Inc}(\pi) = 0$ which means $\exists w \in \Omega$, such that $\pi_1(\omega) = \pi_2(\omega) = 1$. Without losing generality, we assume that $\mathsf{BetP}_{m_2}(w_1) \leq \mathsf{BetP}_{m_1}(w_1)$ (since m_1 and m_2 are symmetric) and we can also assume that there exists a subset A such that $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} = \mathsf{BetP}_{m_2}(A) - \mathsf{BetP}_{m_1}(A)$ holds (otherwise if $\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} = \mathsf{BetP}_{m_1}(A') - \mathsf{BetP}_{m_2}(A')$, we let $A = \Omega \setminus A'$ and the equation still holds). Let the sets of focal elements for m_2 be A_1, .., A_p where $A_1 \subset A_2 \subset ... \subset A_p$ and let $A'_p = A_p \setminus \{\omega_1\}$, we get

$$\begin{aligned}
\mathsf{BetP}_{m_2}(A'_p) &= \frac{|A_1|-1}{|A_1|} m_2(A_1) + ... + \frac{|A_p|-1}{A_p} m_2(A_p) \\
&\leq \frac{|A_p|-1}{A_p} m_2(A_1) + ... + \frac{|A_p|-1}{A_p} m_2(A_p) \text{(see Equation 2)} \\
&= \frac{|A_p|-1}{A_p} (m_2(A_1) + ... + m_2(A_p)) \\
&\leq \frac{|A_p|-1}{A_p},\ \text{(since } m_2(A_1) + ... + m_2(A_p) \leq 1) \\
&\leq \frac{n-1}{n} \text{ (because } A_p \subseteq \Omega\text{, where } |\Omega| = n)
\end{aligned} \tag{3}$$

Then $\mathsf{BetP}_{m_2}(A'_p)$ is the largest value possible among all $\mathsf{BetP}_{m_2}(B)$ where $B \subseteq \Omega \setminus \{\omega_1\}$.

When $w_1 \in A$, we have

$$\begin{aligned}
\mathsf{difBetP}_{\mathsf{m_1}}^{\mathsf{m_2}} &= \mathsf{BetP}_{m_2}(A) - \mathsf{BetP}_{m_1}(A) \\
&= \mathsf{BetP}_{m_2}(A \setminus \{w_1\}) + \mathsf{BetP}_{m_2}(\{w_1\})
\end{aligned}$$

$$
\begin{aligned}
&-\mathsf{BetP}_{m_1}(A \setminus \{w_1\}) - \mathsf{BetP}_{m_1}(\{w_1\}) \\
&\leq \mathsf{BetP}_{m_2}(A \setminus \{w_1\}) - \mathsf{BetP}_{m_1}(A \setminus \{w_1\}) \\
&\leq \mathsf{BetP}_{m_2}(A_p^{'}) - 0 \leq \frac{n-1}{n} \qquad (4)
\end{aligned}
$$

When $w_1 \notin A$, $\mathsf{difBetP}_{\mathsf{m}_1}^{\mathsf{m}_2} = \mathsf{BetP}_{m_2}(A) - \mathsf{BetP}_{m_1}(A) \leq \mathsf{BetP}_{m_2}(A_p^{'}) - 0 \leq \frac{n-1}{n}$. That is, $\mathsf{difBetP}_{\mathsf{m}_1}^{\mathsf{m}_2} \leq \frac{n-1}{n}$ is true for any two possibility distributions.

$\diamond$

This proposition is important, since it tells us that two apparently totally consistent possibility distributions can be very different when we measure their distances between betting commitments to subsets. This means that the two distributions can have very different degrees of possibility assigned to some elements, though they totally agree on some other elements. Therefore, using $\mathsf{Inc}(\pi_\wedge)$ alone may not be accurate enough when assessing how consistent (close) that two possibility distributions are.

Proposition 6. *Let π_1 and π_2 be two possibility distributions and normal, and let their conjunctively combined possibility distribution be $\pi_\wedge$. Furthermore, let m_1 and m_2 be their corresponding mass functions. When $\mathsf{Inc}(\pi_\wedge) = \epsilon$ where ϵ is sufficiently large (like 0.8), we have*

$$\mathsf{difBetP}_{\mathsf{m}_1}^{\mathsf{m}_2} \geq 2\epsilon - 1$$

Proof First we assume that the values of $\pi_1(\omega)$ for all $\omega \in \Omega$ are arranged as (see Definition 4)

$$1 \geq \alpha_1 \geq ... \geq \alpha_i ... \geq \alpha_n > 0$$

Let α_i be the smallest value in the above sequence such that $\alpha_i > 1 - \epsilon$, based on Definition 4, we have a focal element A_i for m_1 as

$$A_i = \{w | \pi_1(w) \geq \alpha_i\}$$

If the other focal elements obtained before A_i are A_1, ..., A_{i-1}, then according to Definition 4, we have

$$\Sigma_{j=1}^{i} m_1(A_j) = (1-\alpha_1) + (\alpha_1 - \alpha_2) + ... + (\alpha_{i-1} - \alpha_i) = 1 - \alpha_i > 1 - (1-\epsilon) = \epsilon$$

By Definition 2, we have $\mathsf{BetP}_{m_1}(A_i) \geq \Sigma_{j=1}^{i} m_1(A_j) > \epsilon$, since $A_i \subset A_{i+1}, ..., A_i \subset A_n$ where $A_{i+1}, ..., A_n$ are the remaining focal elements for m_1.

Similarly, for π_2, there is a subset B_j such that $\mathsf{BetP}_{m_2}(B_j) > \epsilon$. Because $\mathsf{Inc}(\pi_\wedge) = \epsilon$, $A_i \cap B_j = \emptyset$ must hold (otherwise, there is a $\omega \in A_i \cap B_j$ where $\min(\pi_1(\omega), \pi_2(\omega)) > 1 - \epsilon$, so $\mathsf{Inc}(\pi_\wedge) < \epsilon$ which contradict the original assumption).

Given that BetP_{m_2} is a probability function, $\mathsf{BetP}_{m_2}(A_i) \leq 1 - \epsilon$ must hold (because $\mathsf{BetP}_{m_2}(B_j) > \epsilon$ and $A_i \cap B_j = \emptyset$). Therefore, we have

$$\mathsf{BetP}_{m_1}(A_i) - \mathsf{BetP}_{m_2}(A_i) \geq \epsilon - (1-\epsilon) = 2\epsilon - 1$$

Since $\mathsf{difBetP}_{\mathsf{m}_1}^{\mathsf{m}_2} \geq \mathsf{BetP}_{m_1}(A_i) - \mathsf{BetP}_{m_2}(A_i)$ (see Definition 3), we have eventually $\mathsf{difBetP}_{\mathsf{m}_1}^{\mathsf{m}_2} \geq 2\epsilon - 1$.

$\diamond$

This proposition is meaningful when $\epsilon \geq 0.5$ and it states that the distance between betting commitments increases along with the increase of the degree of inconsistency.

4 Merging Operators Selection Criteria

4.1 Merging operators in possibility theory

The fundamental classes of merging operators in possibility theory (or possibilistic logic) are **conjunctive** and **disjunctive** operators. Typical conjunctive operators are *minimum* ($\mathsf{min}(\pi_1(\omega), \pi_2(\omega))$), *product* ($\pi_1(\omega) \times \pi_2(\omega)$), and *linear product* ($\mathsf{max}(\pi_1(\omega) + \pi_2(\omega) - 1, 0)$), and their dual are the *maximum*, the *probabilistic sum* ($\pi_1(\omega) + \pi_2(\omega) - \pi_1(\omega)\pi_2(\omega)$), and the *bounded sum* ($\mathsf{min}(1, \pi_1(\omega)\pi_2(\omega))$). All these conjunctive and disjunctive operators are associative, so merging n possibility distributions can be done recursively, provided that there are no normalizations for the intermediate merging results.

Since some of these operators have special characteristics, two specialized classes of merging operators are further defined in [BDKP02], they are respectively **idempotent** and **reinforcement** operators. For example, the product and the linear product operators are also reinforcement operators, and minimum and maximum are idempotent operators. Furthermore, some **adaptive** operators were proposed which aim at integrating both conjunctive and disjunctive operators when neither of them alone is suitable for merging.

As discussed in [BDKP02], these five classes of operators are suitable for different situations. The conjunctive operators are used when it is believed that all the sources are reliable and these sources agree with each other. When there is a strong agreement among the sources, reinforcement operators are more suitable. On the other hand, the disjunctive operators are applied when it is believed that some sources are reliable but it is not known which of these sources are and when there is a high degree of conflict among sources. Idempotent operators can deal with redundant information where repeated information is only counted once. Since disjunctive operators are too cautious for merging sources with a low level of inconsistency, adaptive operators are suggested to integrate the behaviour of both conjunctive and disjunctive operators.

Among the three named conjunctive operators, it is well recognized ([DP01]) that the product operator is equivalent to the Dempster's combination rule for the computation of the plausibility of singletons. Therefore, the condition of applying Dempster's rule shall apply to this operator as well, i.e., the information comes from distinct or independent sources.

Although the above analysis provides a general guideline as which operator is suitable for what situation, there are no quantitative measures judging precisely when a particular operator should be selected. For example, what value of inconsistency is regarded as *a lower degree* of inconsistency?

We are interested in whether it is possible to provide some quantitative approaches to serving this purpose based on properties we have shown in the previous section, and hence we propose the following guidelines to recommend how to select a merging operator.

4.2 Merging operators selection criteria

In the following, we use $\times$ and $\otimes$ to denote the *product* and the *linear product* operators.

Definition 5. *Let π_1 and π_2 be two possibility distributions and m_1 and m_2 be their corresponding mass functions. When $\mathsf{Inc}(\pi_\wedge) = 0$,*

- *if $\mathsf{difBetP}^{\mathsf{m}_1}_{\mathsf{m}_2} = 0$, then operator $\otimes$ is recommended if the information is from independent (distinct) sources; otherwise, operator* min *is recommended,*
- *if $0 < \mathsf{difBetP}^{\mathsf{m}_1}_{\mathsf{m}_2} < \epsilon_1$, then operator $\times$ is recommended if the information is from independent (distinct) sources; otherwise, operator* min *is recommended,*
- *if $\epsilon_1 \leq \mathsf{difBetP}^{\mathsf{m}_1}_{\mathsf{m}_2} < \epsilon_2$, then operator $\times$ can be applied with caution if the information is from independent (distinct) sources; otherwise, operator* min *is recommended,*
- *if $\epsilon_2 \leq \mathsf{difBetP}^{\mathsf{m}_1}_{\mathsf{m}_2}$, then operator* min *is recommended.*

where ϵ_1 is sufficiently small (e.g., 0.3) and ϵ_2 is sufficiently large (e.g., 0.8).

This definition shows that when $\mathsf{Inc}(\pi_\wedge) = 0$, we do not have to arbitrarily choose a conjunctive operator, the $\mathsf{difBetP}^{\mathsf{m}_1}_{\mathsf{m}_2}$ value provides additional information as whether a high reinforcement operator is more suitable. For example, when $\mathsf{difBetP}^{\mathsf{m}_1}_{\mathsf{m}_2} = 0$, it is more advisable to use $\otimes$ than $\times$ because the information is highly consistent. As stated in [DP94], the condition of choosing such a reinforcement operator is the independence of sources of information. When this condition cannot be guaranteed, min would be a safer option to use.

Definition 6. *Let π_1 and π_2 be two possibility distributions and m_1 and m_2 be their corresponding mass functions. When $0 < \mathsf{Inc}(\pi_\wedge) < \epsilon$,*

- *if $\mathsf{difBetP}^{\mathsf{m}_1}_{\mathsf{m}_2} < \epsilon_1$, then operator $\times$ is recommended if the information is from independent (distinct) sources; otherwise, operator* min *is recommended,*
- *if $\epsilon_1 \leq \mathsf{difBetP}^{\mathsf{m}_1}_{\mathsf{m}_2} < \epsilon_2$, then operator $\times$ can be applied with caution if the information is from independent (distinct) sources; otherwise, operator* min *is recommended,*
- *if $\epsilon_2 \leq \mathsf{difBetP}^{\mathsf{m}_1}_{\mathsf{m}_2}$, then operator* min *is recommended.*

where ϵ is sufficiently small (e.g., 0.2), and ϵ_1 and ϵ_2 are as defined in Definition 5.

Example 4. *let two pairs of possibility distributions be as given below.*

$$\pi^1_1(\omega_1) = 0.7, \pi^1_1(\omega_2) = 0.8, \pi^1_1(\omega_3) = 1.0, \pi^1_1(\omega_4) = 0.6;$$

$$\pi^1_2(\omega_1) = 1.0, \pi^1_2(\omega_2) = 0.9, \pi^1_2(\omega_3) = 0.7, \pi^1_2(\omega_4) = 0.6.$$

and

$$\pi^2_1(\omega_1) = 0.1, \pi^2_1(\omega_2) = 0.8, \pi^2_1(\omega_3) = 1.0, \pi^2_1(\omega_4) = 0.1;$$

$$\pi^2_2(\omega_1) = 1.0, \pi^2_2(\omega_2) = 0.9, \pi^2_2(\omega_3) = 0.2, \pi^2_2(\omega_4) = 0.1.$$

We use $\pi^1_\wedge$ and $\pi^2_\wedge$ to denote the combined possibility distributions from the two pairs using min*. Their degrees of inconsistency are the same, $\mathsf{Inc}(\pi^1_\wedge) = \mathsf{Inc}(\pi^2_\wedge) = 0.2$ and this value suggests the application of a conjunctive operator based on Definition 6.*

The corresponding mass functions from the two pairs of possibility distributions are

$$m^1_1(\{\omega_3\}) = 0.2, m^1_1(\{\omega_2, \omega_3\}) = 0.1, m^1_1(\{\omega_1, \omega_2, \omega_3\}) = 0.1, m^1_1(\{\Omega\}) = 0.6;$$

$$m_2^1(\{\omega_1\}) = 0.1, m_2^1(\{\omega_1, \omega_2\}) = 0.2, m_2^1(\{\omega_1, \omega_2, \omega_3\}) = 0.1, m_2^1(\{\Omega\}) = 0.6;$$

and

$$m_1^2(\{\omega_3\}) = 0.2, m_1^2(\{\omega_2, \omega_3\}) = 0.7, m_1^2(\{\Omega\}) = 0.1;$$

$$m_2^2(\{\omega_1\}) = 0.1, m_2^2(\{\omega_1, \omega_2\}) = 0.7, m_2^2(\{\omega_1, \omega_2, \omega_3\}) = 0.1, m_2^2(\{\Omega\}) = 0.1.$$

For the first pair of mass functions, we have $\mathsf{difBetP}_{\mathsf{m}_1^1}^{\mathsf{m}_2^1} = 0.25$*, while for the 2nd pair we get* $\mathsf{difBetP}_{\mathsf{m}_1^2}^{\mathsf{m}_2^2} = 0.525$*. These two pairs show an obvious difference in* $\mathsf{difBetP}$ *values. The possibility distributions in the first pair are more consistent with each other than the two in the 2nd pair. However, this information is not reflected by* $\mathsf{Inc}(\pi_\wedge)$*.*

According to Definition 6, the first pair can be combined with the product operator $(\times)$ *if the sources of information are distinct while it is better to merge the second pair with the minimum operator.*

Definition 7. *Let* π_1 *and* π_2 *be two possibility distributions and* m_1 *and* m_2 *be their corresponding mass functions. When* $\mathsf{Inc}(\pi_\wedge) \geq \epsilon$ *then a disjunctive operator is recommended to merge* π_1 *and* π_2*, where* ϵ *is sufficiently large, e.g.,* 0.8*.*

Example 5. Let two possibility distributions be

$$\pi_1(\omega_1) = 0.1, \pi_1(\omega_2) = 0.2, \pi_1(\omega_3) = 1.0, \pi_1(\omega_4) = 0.1;$$

$$\pi_2(\omega_1) = 1.0, \pi_2(\omega_2) = 0.2, \pi_2(\omega_3) = 0.2, \pi_2(\omega_4) = 0.1.$$

Let $\pi_\wedge$ *be the possibility distribution combining* π_1 *and* π_2 *with* min*, then the degree of inconsistency is* $\mathsf{Inc}(\pi_\wedge) = 0.8$ *which suggests a high degree of inconsistency. Therefore, the conjunctive operators are unlikely to be used.*

On the other hand, the two corresponding mass functions from the possibility distributions are

$$m_1(\{\omega_3\}) = 0.8, m_1(\{\omega_2, \omega_3\}) = 0.1, m_1(\{\Omega\}) = 0.1;$$

$$m_2(\{\omega_1\}) = 0.8, m_2(\{\omega_1, \omega_2, \omega_3\}) = 0.1, m_2(\{\Omega\}) = 0.1;$$

and $\mathsf{difBetP}_{\mathsf{m}_1}^{\mathsf{m}_2} = 0.72$ *which also hints a strong conflict among the two pieces of information.*

The *grey area* that the above three definitions did not cover is when $\epsilon_1 \leq \mathsf{Inc}(\pi_\wedge) \leq \epsilon_2$, such that $\epsilon_1 = 0.3, \epsilon_2 = 0.8$. In our future work, we will further investigate what other measures are needed in order to select a suitable merging operator for this situation.

5 Conclusion

In this paper, we have shown that additional approaches to measuring conflict among pieces of uncertain information are needed since the only measure used in possibility theory, e.g., the degree of inconsistency, is not sufficient.

We have studied how the distance between betting commitments developed in DS theory can be used to measure the inconsistency among pieces of uncertain information in possibility theory. We have also established a set of properties to show the relationship between the degree of inconsistency and the distance between betting commitments between a pair of uncertain information. We conclude that these two measures tell us different aspects of the information and both values should be used to select a suitable merging operator. This investigation can be taken as the refinement of general discussions on merging operators selection in [BDKP02].

As pointed out in the paper, there is a *grey area* where it is not clear which merging operator is best suited. One of our future work is to explore other additional measures to see if some quantitative measures can be proposed to deal with these cases.

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