On the Merit of Selecting Different Belief Merging Operators

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Abstract. Belief merging operators combine multiple belief bases (a profile) into a collective one. When the conjunction of belief bases is consistent, all the operators agree on the result. However, if the conjunction of belief bases is inconsistent, the results vary between operators. There is no formal manner to measure the results and decide on which operator to select. So, in this paper we propose to evaluate the result of merging operators by using three ordering relations (fairness, satisfaction and strength) over operators for a given profile. Moreover, a relation of conformity over operators is introduced in order to classify how well the operator conforms to the definition of a merging operator. By using the four proposed relations we provide a comparison of some classical merging operators and evaluate the results for some specific profiles.

1 Introduction

Belief merging looks at strategies for combining belief bases from different sources, which in conjunction may be inconsistent, in order to obtain a consistent belief base representing the group. Logic-based belief merging has been studied extensively [1, 14, 10, 11, 8, 9]. A well known strategy is the use of an operator Δ which takes as input the belief bases (profile) E and outputs a new consistent merged belief base $\Delta(E)$. Often operators require additional information such as a priority relationship between the bases or numbers representing base weights. However, in many applications this information does not exist and we must accord equal importance to each of the beliefs and bases. Among existing operators which are independent of additional information, we can mention: Δ_{MCS}, Δ_{Σ}, Δ_{Gmax} and DA^2 operators. In each case the belief bases are described using a finite number of propositional symbols; there is no hierarchy, nor priority, nor any difference in reliability of the sources. Prioritized belief bases or weighted bases, such as in possibilistic logic [12], will be consider in future work.

Considering flat profiles, there are two main families of belief merging operators: the formula-based operators (also called syntax-based operators) and the model-based operators (also called distance-based operators). The operators belonging to the former family select subsets of consistent formulae from the profile E. While the variety depends on the selection criterion, there is no formal way to compare different criteria. The operators belonging to the latter family define

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a distance between worlds, a distance from worlds to bases, and a distance from worlds to profiles with the help of an aggregation function. Then, the operators take as models of the merged result, those worlds which are closest to the belief profile. While the distances allow us to define a notion of closeness in any framework, we miss a general measure that indicates how close a profile E is to the merged base $\Delta(E)$. A general measure will allow us to compare, in a formal manner, the results from different operators.

In [2] the notion of a base satisfaction index (individual index) was introduced, where such an index measures the closeness from a base K to a merged base $\Delta(E)$. The index is a function that takes as input two belief bases: a belief base $K \in E$ and the merged base $\Delta(E)$. The output is a numeral value $i(K, \Delta(E))$ which represents the degree of satisfaction of the base given the merged base. While this notion allows us to measure the satisfaction of every member of a profile, there is no measure for the satisfaction of the whole profile. So, in this paper we propose to evaluate the result of merging operators by using three ordering relations (fairness, satisfaction and strength) over the operators for a given profile. Moreover, a relation of conformity over the operators is introduced in order to classify the degree to which an operator conforms to the definition of a merging operator. By using the four proposed relations we provide a comparison for some classical merging operators and we evaluate the results of these operators for some specific profiles.

The objective of this paper is to draw a comparative landscape through criterion based on a degree of satisfaction and notions of conformity and strength for many merging operators from the literature. We focus on operators for merging bases represented as sets of propositional formulae, where no priorities or weights are given. The rest of the paper is organized as follows. After some preliminaries, in Section 3 we recall some of the main merging operators from the literature. Then, we introduce the degree of satisfaction and three relations as well as introducing a method for measuring the result of the operators. In Section 5 we compare some results for operators through our proposal for specific profiles. Finally, we conclude mentioning some future work.

2 Preliminaries

We consider a language $\mathcal{L}$ of propositional logic using a finite set of propositional variables $P := \{p_1, p_2, ..., p_m\}$, the standard connectives, and the boolean constants $\top$ and $\bot$ representing always $true$ and $false$, respectively. $|A|$ denotes the cardinality of a set A or the absolute value of a number A.

An interpretation or world w is a function from P to $\{0, 1\}$, the set of worlds of the language is denoted by $\mathcal{W}$, its elements will be denoted by boolean vectors of the form $(w(p_1), ..., w(p_m))$, where $w(p_i) = 1$ (representing $true$) or $w(p_i) = 0$ (representing $false$) for $i = 1, ..., m$. A world w is a model of $\phi \in \mathcal{L}$ if and only if ϕ is true under w in the classical truth-functional manner. The set of models of a formula ϕ is denoted by $mod(\phi)$. The formula ϕ is consistent if and only if there exists a model of ϕ. The formula ϕ is a logical consequence of a formula ψ, denoted $\psi \models \phi$ if and only if $mod(\psi) \subseteq mod(\phi)$. For any set of models $M \subseteq \mathcal{W}$, let $form(M)$ denote a formula whose set of models are precisely M (up to logical equivalence), i.e., $mod(form(M)) = M$.

A *belief base* K is a finite set of propositional formulae of $\mathcal{L}$ representing the beliefs from a source. Some approaches identify K by the conjunction of its elements so each knowledge base can be treated as a single formula. For this reason we use L rather than $2^{\mathcal{L}}$ or $\mathcal{L}$ to denote the set of all belief bases. A belief profile E is a multiset (bag) of n belief bases $E = \{K_1, ..., K_n\}$ ($n \geq 1$). The profile represents the set of information sources to be processed. We denote the conjunction of bases in E by $\bigwedge E$ and the disjunction of bases in E by $\bigvee E$. A profile E is consistent if and only if $\bigwedge E$ is consistent. The multi-set union between E_1 and E_2 is denoted by $E_1 \sqcup E_2$.

In [6] eight postulates have been proposed to characterize the process of belief merging with integrity constraints in a propositional setting. This characterization is rephrased, without reference to integrity constraints, producing the following M1–M6 postulates.

Definition 1. *Let E, E_1, E_2 be belief profiles, K_1 and K_2 be consistent belief bases. Let Δ be an operator which assigns to each belief profile E a belief base $\Delta(E)$. Δ is a merging operator if and only if it satisfies the following postulates:*

(M1) $\Delta(E)$ *is consistent*
(M2) if $\bigwedge E$ *is consistent then* $\Delta(E) \equiv \bigwedge E$
(M3) if $E_1 \equiv E_2$*, then* $\Delta(E_1) \equiv \Delta(E_2)$
(M4) $\Delta(\{K_1, K_2\}) \wedge K_1$ *is consistent if and only if* $\Delta(\{K_1, K_2\}) \wedge K_2$ *is consistent*
(M5) $\Delta(E_1) \wedge \Delta(E_2) \models \Delta(E_1 \sqcup E_2)$
(M6) if $\Delta(E_1) \wedge \Delta(E_2)$ *is consistent, then* $\Delta(E_1 \sqcup E_2) \models \Delta(E_1) \wedge \Delta(E_2)$

The postulates describe the principles that a belief merging operator should satisfy. Among them, syntax irrelevance (M3), and fairness (M4) are key postulates. In the literature [6] we can find some operators which are considered merging operators even though they do not satisfy the six postulates. Therefore a relation, based on the number of postulates for which operators conform, may be a first attempt at comparing operators. Formally:

Conformity relation. An operator $\mathit{\Delta}_1$ is *more conforming than* an operator $\mathit{\Delta}_2$, denoted $\mathit{\Delta}_1 \geq \mathit{\Delta}_2$, if $\mathit{\Delta}_1$ satisfies more postulates than $\mathit{\Delta}_2$.

This relation is strictly numeric in that we do not consider the satisfaction of any one postulate to be more desirable than the satisfaction of another.

3 Belief merging operators

As we stated before, there are two main families of merging operators: formula-based and model-based operators. The former selects some formulae from the union of the bases with the help of a selection criterion. The latter selects some interpretation with the help of some distances and aggregation functions.

3.1 Formula-based operators

Formula-based operators are based on the selection of consistent subsets of formulae in the union of the members of a profile E. In [1], the operators aim to find all maximally consistent subsets (MCS) of the inconsistent union of belief bases. When an integrity constraint is imposed the operator only selects the MCS which are consistent w.r.t. the integrity constraint. The operators are defined w.r.t. a function MCS, whose input is a belief base K and an integrity constraint μ, and the output is the set of maximal (w.r.t. inclusion) consistent subsets of $K \cup \{\mu\}$ that contains μ, formally, $MCS(K, \mu)$ is the set of all F s.t.:

1. F is consistent, 2. $F \subseteq K \cup \{\mu\}$,
3. $\mu \in F$ and 4. if $F \subset F' \subseteq K \cup \{\mu\}$, then F' is inconsistent.

MCS is extended for a profile E as follows: $MCS(E, \mu) = MCS(\bigcup_{K \in E} K, \mu)$.

Another function that helps to define some operators is $|MCS|$, which can be defined by replacing inclusion with cardinality in 4: if $|F| < |F'|$, s.t. $F' \subseteq K \cup \{\mu\}$, then F' is not consistent. $|MCS|$ is extended for a profile E in a similar manner. The following operators have been defined in [1, 5]:

1. $\Delta_{MCS_1}(E, \mu) = \bigvee MCS(E, \mu)$.
2. $\Delta_{MCS_3}(E, \mu) = \bigvee\{F : F \in MCS(E, \top) \text{ and } F \cup \{\mu\} \text{ consistent}\}$.
3. $\Delta_{MCS_4}(E, \mu) = \bigvee |MCS|(E, \mu)$.
4. $\Delta_{MCS_5}(E, \mu) = \begin{cases} \bigvee \{F \cup \{\mu\} : F \in MCS(E, \top) & \text{if } \exists F \in MCS(E, \top) \\ \text{and } F \cup \{\mu\} \text{ consistent}\} & \text{s.t. } F \cup \{\mu\} \neq \emptyset, \\ \mu & \text{otherwise.} \end{cases}$

The first three operators correspond respectively to operators $Comb1(E, \mu)$, $Comb3(E, \mu)$ and $Comb4(E, \mu)$ proposed in [1]. In order to assure consistency, Δ_{MCS_3} was modified in [5] as Δ_{MCS_5}. These operators are syntax sensitive.

Example 1. From [11]. Let $E = \{K_1, K_2, K_3\}$ where $K_1 = \{a\}$, $K_2 = \{a \rightarrow b\}$ and $K_3 = \{a, \neg b\}$. Then, $\Delta_{MCS}(E) = \{a, a \rightarrow b\} \vee \{a, \neg b\} \vee \{\neg b, a \rightarrow b\}$.

3.2 Model-based operators

In most model-based frameworks an operator Δ is defined by a function $m : L^n \rightarrow 2^{\mathcal{W}}$ from the set of profiles to the power set of $\mathcal{W}$ s.t. $\Delta(E) = form(m(E))$. For simplicity we use the standard notation $mod(\Delta(E))$ rather than $m(E)$. The process is defined using three distances: a distance from one world to another $d(w, w')$, a distance from a world to a belief base $d(w, K)$ based on $d(w, w')$ and a distance from a world to a profile $d(w, E)$ based on $d(w, K)$. The latter distance is usually defined by aggregation functions and allows us to define a pre-order $\leq_E$. The closest worlds to the profile are the models of the merging process.

We summarize the definitions as follows. A distance[1] between worlds is a function $d : \mathcal{W} \times \mathcal{W} \rightarrow \mathbb{R}^+$ from the Cartesian square of $\mathcal{W}$ to the set of positive real numbers s.t. for all $w, w' \in \mathcal{W}$:

$$1.\ d(w, w') = d(w', w) \quad \text{and} \quad 2.\ d(w, w') = 0 \text{ iff } w = w'.$$

[1] As in [9], the triangle inequality is not required.

The distance between a world and a belief base is a function $d : \mathcal{W} \times L \rightarrow \mathbb{R}^+$ from the Cartesian product of $\mathcal{W}$ and the set of belief bases to the set of positive real numbers. Some methods define this distance as the minimal distance between world w and any model of base K, i.e., $d(w, K) = min_{w' \in mod(K)} d(w, w')$. Finally, the distance between a world and a profile is a function $d_a : \mathcal{W} \times L^n \rightarrow \mathbb{R}^+$ from the Cartesian product of $\mathcal{W}$ and the set of profiles to the set of positive real numbers, defined as the result of applying the aggregation function $a : \mathbb{R}^{+^n} \rightarrow \mathbb{R}^+$ to the distances between w and every profile member, i.e. $d_a(w, E) = a(d(w, K_1), ..., d(w, K_n))$ s.t. $E = \{K_1, ..., K_n\}$.

Definition 2. *An aggregation function a is a total function associating a positive real number to every finite n-tuple of positive real numbers s.t. for all $x_1, ..., x_n, x, y \in \mathbb{R}^+$:*

1. *if $x \leq y$, then $a(x_1, ..., x, ..., x_n) \leq a(x_1, ..., y, ..., x_n)$,*
2. *$a(x_1, ..., x_n) = 0$ iff $x_1 = ... = x_n = 0$ and*
3. *$a(x) = x$.*

Any aggregation function induces a total pre-order $\leq_E$ on the set $\mathcal{W}$ w.r.t. the distances to a given profile E. Thus, the merging operator $\Delta_{d,a}$ for a profile E is defined as a belief base (up to logical equivalence) whose models are the set of all worlds with the minimal distance d_a to the profile E, i.e.,

$$mod(\Delta_{d,a}(E)) = min(\mathcal{W}, \leq_E).$$

Every framework consists of a distance and an aggregation function. The distance between worlds most widely used in the literature is Hamming distance[2], which is the number of propositional variables on which two worlds differ, i.e.,

$$d(w, w') = \sum_{p \in P} |w(p) - w'(p)|.$$

Two outstanding aggregation functions are maximum and sum, their corresponding distance are defined, respectively, as follows:

$$d_{max}(w, E) = \max_{K \in E} d(w, K) \quad \text{and} \quad d_{\Sigma}(w, E) = \sum_{K \in E} d(w, K).$$

In both cases, the induced pre-order is defined with the help of $\leq$ over real numbers as follows:

$$w \leq_E w' \quad \textit{iff} \quad d_a(w, E) \leq d_a(w', E).$$

Another well known operator is Δ_{Gmax}, introduced in [7], where the aggregation function does not output a number but instead outputs a vector of numbers, which is the result of sorting the input distances in descending order, i.e.,

$$d_{Gmax}(w, E) = sort(d(w, K_1), ..., d(w, K_n)).$$

[2] From now, if a belief merging operator $\Delta_{d,a}$ uses Hamming distance, in order to avoid heavy notations, we identify it by Δ_a.

The operator Δ_{Gmax} uses the lexicographic ordering $\leq_{lex}$ for comparing vectors, the pre-order induced is defined as follows.

$$w \leq_E w' \quad \textit{iff} \quad d_{Gmax}(w, E) \leq_{lex} d_{Gmax}(w', E).$$

Example 2. Given profile E from Example 1, and variables a and b in that order, then: $mod(\Delta_{max}(E)) = \{(0,0),(1,0),(1,1)\}$; $mod(\Delta_{\Sigma}(E)) = \{(1,0),(1,1)\}$; and $mod(\Delta_{Gmax}(E)) = \{(1, 0),(1,1)\}$.

4 On the measure of merging operators

Comparing the number of postulates (from Definition 1) for which merging operators conform, provides a means to generally evaluate and compare operators. Only Δ_{Σ} and Δ_{Gmax} satisfy all six postulates. Δ_{max} satisfies the first five postulates. However, given a naive operator $\Delta_{\top}$ (typical of Yager's rule for merging belief functions [15]) s.t. $\Delta_{\top} = \bigwedge E$ if $\bigwedge E$ is consistent and $\Delta_{\top} = \top$ otherwise. This operator satisfies the first five postulates and satisfies the last postulate when both profiles E_1 and E_2 are either consistent or inconsistent. Under this characterization we can consider $\Delta_{\top}$ to be more conforming than operators such as Δ_{max} which satisfy fewer postulates. However $\Delta_{\top}$ does not help to make decisions since the result is a tautology when the sources of information are inconsistent and the information of a tautology is neither useful nor informative. For this reason, we also need to classify operators based on their merging result in order to select the best operator for a given profile.

We propose to classify operators based on: (1) conformity; (2) the degree of satisfaction of their merging result w.r.t the given profile and two relations over operators; and (3) a relation of strength over operators. The degree of satisfaction of a belief base is formally defined as follows:

Definition 3 (Degree of satisfaction of belief bases). *Function SAT : $L \times L \to [0,1]$ is called a the degree of satisfaction of belief bases iff for any belief base K and K', it satisfies the following postulates:*

Reflexivity: $SAT(K, K') = 1$ *iff* $mod(K') \cap mod(K) \neq \emptyset$.
Monotonicity: $SAT(K, K') \geq SAT(K, K^*)$ *iff* $mod(K') \subseteq mod(K^*)$.

Semantically, a degree of satisfaction for a belief base K in a given profile E, is a measure of how satisfied the belief base K is by the merged base $K' = \Delta(E)$ resulting from the application of a merging operator Δ on the profile E. Notice that the definition considers a general case where K' may be a belief base which is not necessarily the result of a merging operator. Two stronger variants are:

Definition 4. *A rational degree of satisfaction is a degree of satisfaction which satisfies the Rationality postulate: $SAT(K, K') = 0$ if $mod(K') \cap mod(K) = \emptyset$.*

Definition 5. *A symmetric degree of satisfaction is a degree of satisfaction which satisfies the Symmetry postulate: $SAT(K, K') = SAT(K', K)$.*

Based on this degree of satisfaction we can define the degree of satisfaction of a profile as follows:

Definition 6 (Degree of satisfaction of belief profiles). *Let E be a profile, SAT be a degree of satisfaction of belief bases and a be an aggregation function. The degree of satisfaction of E by K' based on SAT and a, denoted $SAT_a(E, K')$, is defined as follows: $SAT_a(E, K') = a_{K \in E} SAT(K, K')$.*

Then we can define a maximum and minimum degree of satisfaction for a profile E as follows:

Definition 7. *Let E be a profile and K' be a belief base. Then $SAT_{max}(E, K')$ is the maximum degree of satisfaction of E by K' iff $SAT_{max}(E, K') = max_{K \in E}$ $SAT(K, K')$. Also, $SAT_{min}(E, K')$ is the minimum degree of satisfaction of E by K' iff $SAT_{min}(E, K') = min_{K \in E} SAT(K, K')$.*

4.1 Instantiation of the degree of satisfaction (base satisfaction index)

Notice that Definition 3 is about properties of a measure, no specific measures are actually given. This section and the next provide these measures. In the literature we can find a way to define the satisfaction of a base given the merged base: in [2] the notion of a base satisfaction index is the degree of satisfaction of $K \in E$, given $\Delta(E)$, as a total function i from $L \times L$ to $[0, 1]$. Then $i(K, \Delta(E))$ indicates how close a base K is to the merged base $\Delta(E)$. In [2] four indexes are proposed when no additional information about the sources is available: i_w, i_s, i_p and i_d. These base satisfaction indexes satisfy Definition 3, so they can be considered as degrees of satisfaction of a belief base[3]. Formally:

Definition 8 (weak drastic index). *This boolean index takes value 1 if the merging result is consistent with the base and 0 otherwise, formally:*

$$i_w(K, \Delta(E)) = \begin{cases} 1 & \textit{if } K \wedge \Delta(E) \textit{ is consistent,} \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 9 (strong drastic index). *This boolean index takes value 1 if the belief base is a logical consequence of the merging result and 0 otherwise, formally:*

$$i_s(K, \Delta(E)) = \begin{cases} 1 & \textit{if } \Delta(E) \models K, \\ 0 & \textit{otherwise.} \end{cases}$$

Definition 10 (probabilistic index). *This index takes the value of the probability of getting a model of K among the models of $\Delta(E)$, formally:*

$$i_p(K, \Delta(E)) = \begin{cases} 0 & \textit{if } |mod(\Delta(E))| = 0 \\ \frac{|mod(K) \cap mod(\Delta(E))|}{|mod(\Delta(E))|} & \textit{otherwise.} \end{cases}$$

[3] For the sake of readability, we use 'base satisfaction index' and 'degree of satisfaction of a belief base' as synonyms, however, notice that a belief satisfaction index was defined without imposing properties.

So, i_p takes its minimal value 0 when no model of K is in the models of the merged base $\Delta(E)$ and its maximal value when each model of the merged based is a model of K. The fact that i_p is based on model counting allows some granularity in the notion of satisfaction. Notice, i_s can be obtained by truncating or dropping the decimal numbers of the i_p result. In fact, i_p can be seen as the probability of getting the belief base as a logical consequence of the merged result.

Definition 11 (Dalal index). *This index grows antimonotonically with the Hamming distance between the two bases under consideration, i.e., the minimal distance between a model of the base K and a model of base $\Delta(E)$, formally:*

$$i_d(K, \Delta(E)) = 1 - \frac{\min_{w \in mod(K), w' \in mod(\Delta(E))} d(w, w')}{|P|}.$$

This index takes its minimal value when every variable must be flipped to obtain a model of $\Delta(E)$ from a model of K, while takes its maximal value whenever K is consistent with $\Delta(E)$ and no flip is required.

Examples for these indexes are shown in Tables 2 and 3. We can propose other indexes such as $i'_p = \frac{|mod(K) \cap mod(\Delta(E))|}{|mod(\Delta(K))|}$: the probability of getting the merged result as a logical consequence of the belief base. However, there is no background theory to support this proposal. Next we introduce a new base satisfaction index based on inconsistency measures for propositional belief bases [3, 4]. Considering the level in which the inconsistency is measured, there are two classes of measures: Base-level measures and Formula-level measures. Those in the former class measure the inconsistency of the belief base as a whole. While those in the latter class measure the degree to which each formula in the belief base is responsible for the inconsistency of the base. The output of the former is a number while the output of the latter is a numerical vector with elements representing each formula in the belief base. This work considers solely the former class. Another classification found in the literature considers *how* inconsistency is measured. In this case there are two main types of measures: Formula-centric measures that count the number of formulae required for creating the inconsistency: the more formulae required to produce an inconsistency, the less inconsistent the base; and atom-centric measures, that take into account the proportion of the language affected by inconsistency: the more propositional variables affected, the more inconsistent the base.

Definition 12 (Base-level measure of inconsistency). *An inconsistency measure on a belief base is a function $I : L \to \mathbb{R}$.*

Diverse measures are defined in [3, 4], however, we will choose the measures which satisfy two properties: syntax-insensitivity, i.e. the measures of two equivalent belief bases are equal; and normalization, i.e. the measure is a real number between 0 and 1. The former is required in order to assure fairness of evaluation w.r.t. the way of writing formulae. The latter is required to assure uniformity in the evaluation. Moreover, we consider degrees of satisfaction between 0 and 1, representing 0 and 100% satisfaction, respectively. As far as we know the only measure that satisfies both properties is I_{LP_m} [4].

The inconsistency measure I_{LP_m} is defined as the normalized minimum number of inconsistent truth values in the LP_m models of the belief base. Formally:

$$I_{LP_m}(K) = \frac{min_{w \in mod_{LP}(K)}(|w!|)}{|P|}$$

where K is a belief base and LP_m extends the notion of worlds considering three truth values $\{0,\ 1,\ \frac{1}{2}\}$, representing *true*, *false* and the additional truth value *both* meaning both "true and false". Then a world is a function from P to $\{0,\ 1,\ \frac{1}{2}\}$. 3^P is the set of all worlds for LP_m. Truth values are ordered as $0 <_t \frac{1}{2} <_t 1$ and $w(\top) = 1$, $w(\bot) = 0$, $w(\neg\phi) = \frac{1}{2}$ iff $w(\phi) = \frac{1}{2}$, $w(\neg\phi) = 1$ iff $w(\phi) = 0$, $w(\phi \wedge \psi) = min_{\leq_t}(w(\phi), w(\psi))$ and $w(\phi \vee \psi) = max_{\leq_t}(w(\phi), w(\psi))$. The LP_m models of the belief base are defined as: $mod_{LP}(K) = \{w \in 3^P \mid w(K) \in \{1, \frac{1}{2}\}\}$ and $w! = \{x \in P \mid w(x) = \frac{1}{2}\}$. The minimum models of a formula are: $min(mod_{LP}(\phi)) = \{w \in mod_{LP}(\phi) \mid \nexists w' \in mod_{LP}(K) \text{ s.t. } w'! \subset w!\}$.

Definition 13 (Base-level inconsistency index). *The base-level inconsistency index is defined as:* $i_i(K, \Delta(E)) = 1 - I(K \cup \Delta(E))$.

This index grows antimonotonically with the base-level measure of inconsistency I between the union of the two bases under consideration. This index takes its minimal value when the degree of inconsistency of the union of the bases is the maximum, while it takes its maximal value whenever the union of the bases is consistent. We consider only the instance: $i_L(K, \Delta(E)) = 1 - I_{LP_m}(K \cup \Delta(E))$.

Proposition 1. *The five satisfaction indexes are degrees of satisfaction. More specifically* i_s, i_w *and* i_p *are rational degrees of satisfaction. Also* i_d *and* i_L *are symmetric degrees of satisfaction.*

4.2 Instantiation of the general degree of satisfaction (profile satisfaction index)

Using the base satisfaction indexes, one can define a satisfaction index for the whole profile. The profile satisfaction indexes are instantiations of degrees of satisfaction of belief profiles, in this work we will use both notions indistinctly. The notion of a profile satisfaction index is the degree of satisfaction of E, given $\Delta(E)$. The index is defined as a total function i from $L^n \times L$ to $\mathbb{R}$. Thus, $i(E, \Delta(E))$ indicates how close a profile is to the merged base $\Delta(E)$, formally:

Definition 14 (Profile satisfaction index). *Let* E *be a profile,* i *be a base satisfaction index and* a *be an aggregation function, the profile satisfaction index based on* i *and* a *is defined as follows:* $i_a(E, \Delta(E)) = a_{K \in E} i(K, \Delta(E))$.

There are many ways to measure the satisfaction of the profile given the merged base. The following measure says that a profile is as satisfied as the satisfaction of its least satisfied element, it is an instantiation of the minimum degree of satisfaction of a profile.

$$i_{min}(E, \Delta(E)) = min_{K \in E} i(K, \Delta(E))$$

Alternatively, another measure says that a profile is satisfied holistically, as the sum of the satisfaction of its elements.

$$i_{\Sigma}(E, \Delta(E)) = \Sigma_{K \in E} i(K, \Delta(E)).$$

4.3 Evaluating merging operators

Postulate M4 only says that no preference should be given to either belief base if they are inconsistent, however this is questioned in the literature. It is possible for us to define a more refined postulate (relation) of fairness using the degree of satisfaction, s.t. we can assign a relative degree of fairness to an operator.

Fairness relation. An operator Δ_1 is *fairer than* an operator Δ_2, denoted $\Delta_1 \succeq \Delta_2$ iff for all E, $SAT_{max}(E, \Delta_1(E)) - SAT_{min}(E, \Delta_1(E)) \leq SAT_{max}(E, \Delta_2(E)) - SAT_{min}(E, \Delta_2(E))$.

This means that a fairer operator minimizes the difference between degrees of satisfaction among bases. We can also define a satisfaction relation between operators based on the degree of satisfaction SAT as follows:

Satisfaction relation. An operator Δ_1 is *more satisfactory than* an operator Δ_2, denoted $\Delta_1 \sqsupseteq \Delta_2$ if for all E, $SAT_a(E, \Delta_2(E)) \leq SAT_a(E, \Delta_1(E))$.

This means that a more satisfactory operator maximizes the degree of satisfaction of belief profiles. Both types of ordering relations (fairness and satisfaction) can be used to select the best operators in terms of these criterion. See Table 1.

However, operators such as $\Delta_\top$ will have the highest degree of fairness and satisfaction, in comparison to the remaining operators. Moreover, in relation to M1–M6 postulates, the operator $\Delta_\top$ is considered more (or at least equally) conforming than Δ_{MCS} and Δ_{max}. We can conclude that $\Delta_\top$ is a good choice. However, $\Delta_\top$ does not produce useful and informative results since they will be a tautology when the profile is inconsistent, so, the conformity, fairness and satisfaction relations are insufficient. For this reason we need some way to classify the degree of "useful and informative" merging results and so we propose to use the notion of *strength* introduced in [13]. With this notion we can say that $\Delta_\top$ is weaker than the other operators since its merging results are weaker.

Strength relation. An operator Δ_1 is *stronger than* an operator Δ_2, denoted $\Delta_1 \supseteq \Delta_2$, if for all E, $mod(\Delta_1(E)) \subseteq mod(\Delta_2(E))$.

Using this notion, we can conclude that the merging operator Δ_{Gmax} is stronger that Δ_{max} and the operator $\Delta_\top$ is the weakest (see Table 1).

In short, the postulates M1–M6 allow us to define a conformity relation between operators s.t. an operator which satisfies more postulates is considered more conforming. Additionally, a degree of satisfaction allows us to define another relation between operators s.t. an operator with a higher degree of satisfaction is 'better' than an operator with a lower degree of satisfaction, i.e., the operator is closer to the original information in comparison to other possible merging results (assuming different merging operators are available). Based on this degree of satisfaction we define another relation of fairness over operators. Finally, we define a strength relation over operators. Unfortunately, these 4 relations over operators cannot identified the best operator in a general case, i.e. for every profile (see Table 1). However, we can combine the strength relation with the fairness and satisfaction relations to define a method to classify the operators results for a given profile. Notice the relations of fairness, satisfaction and strength can be used for particular cases of E, where we can say, for example, that the operator Δ_1 is stronger than Δ_2 for a given E if $mod(\Delta_1(E)) \subseteq mod(\Delta_2(E))$.

Δ_1 \ Δ_2	Δ_{MCS}	Δ_{max}	Δ_{Σ}	Δ_{Gmax}	$\Delta_{\top}$
Δ_{MCS}	$\geq, \succeq, \sqsupseteq, \supseteq$	n/a	$\not\geq$	$\not\geq$	$\not\geq$
Δ_{max}	$\geq$	$\geq, \succeq, \sqsupseteq, \supseteq$	$\not\geq$	$\not\geq, \sqsupseteq$	$\geq$
Δ_{Σ}	$\geq$	$\geq$	$\geq, \succeq, \sqsupseteq, \supseteq$	$\geq$	$\geq$
Δ_{Gmax}	$\geq$	$\geq, \supseteq$	$\geq$	$\geq, \succeq, \sqsupseteq, \supseteq$	$\geq$
$\Delta_{\top}$	$\geq, \succeq, \sqsupseteq, \not\supseteq$	$\geq, \succeq, \sqsupseteq, \not\supseteq$	$\not\geq, \succeq, \sqsupseteq, \not\supseteq$	$\not\geq, \succeq, \sqsupseteq, \not\supseteq$	$\geq, \succeq, \sqsupseteq, \supseteq$

Table 1. Comparison of operators in terms of operators being more conforming ($\Delta_1 \geq \Delta_2$), fairer ($\Delta_1 \succeq \Delta_2$), more satisfactory ($\Delta_1 \sqsupseteq \Delta_2$) or stronger ($\Delta_1 \supseteq \Delta_2$), where *n/a* means not comparable or not found.

Example 3. From [14, 7]. Let $E = \{K_1, K_2, K_3\}$ where $K_1 = \{(S \vee O) \wedge \neg D\}$, $K_2 = \{(\neg S \wedge D \wedge \neg O) \vee (\neg S \wedge \neg D \wedge O)\}$ and $K_3 = \{S \wedge D \wedge O\}$.

Using i_p for Example 3, we have $i_{p,max}(E, \Delta_{max}(E)) = 0.33$, $i_{p,max}(E, \Delta_{Gmax}(E)) = 1$, and $i_{p,min}(E, \Delta_{max}(E)) = i_{p,min}(E, \Delta_{Gmax}(E)) = 0$ (see Table 3). So, for this E, Δ_{max} is fairer than Δ_{Gmax} but using $i_{p,\Sigma}$, Δ_{Gmax} is more satisfactory than Δ_{max}. Moreover, as stated previously, Δ_{Gmax} is stronger than Δ_{max} and Δ_{Gmax} is more conforming than Δ_{max} since Δ_{Gmax} conforms to all six postulates while Δ_{max} only conforms to five.

Even for a particular E the selection of a "best result" is not always evident. In order to classify operators for any profile we must generalize two relations. For this reason we extend the fairness and satisfaction relations for belief bases rather than for the result of operators , as follows:

Fairness relation over belief bases. A belief base K_1 is *fairer than* a base K_2, denoted $K_1 \succeq K_2$, for every profile E if $SAT_{max}(E, K_1) - SAT_{min}(E, K_2) \leq SAT_{max}(E, K_1) - SAT_{min}(E, K_2)$.

Satisfaction relation over belief bases. A belief base K_1 is *more satisfactory than* a base K_2, denoted $K_1 \sqsupseteq K_2$, for every profile E if $SAT_a(E, K_2) \leq SAT_a(E, K_1)$.

Now, notice that if Δ_1 is stronger than Δ_2 for a given E then there exists a set of worlds Ω s.t. $mod(\Delta_1(E)) \cup \Omega = mod(\Delta_2(E))$, i.e. some worlds appearing in $\Delta_2(E)$ may be 'erased' in the process of merging with Δ_1. If $\Delta_1(E)$ is fairer than $form(\Omega)$ and $\Delta_1(E)$ is more satisfactory than $form(\Omega)$, we can conclude that the worlds which have been 'eliminated' by Δ_1 do not affect the properties of fairness and satisfaction of Δ_1 w.r.t. the extra worlds in $mod(\Delta_2(E))$; and given that Δ_1 is stronger than Δ_2 we can conclude that the result of Δ_1 is better that the result of Δ_2. In selecting a result, we can say that Δ_1 offers less choice than Δ_2 and so it is more useful for making decisions.

5 Comparing operators results

In this section we demonstrate instantiations of the degrees of satisfaction (i_w, i_s, i_p, i_d and i_L) and their corresponding satisfaction profile indexes as applied to two profiles selected from the literature. Satisfaction indexes for Example 1 (resp. Example 3) are shown in Table 2 (resp. Table 3).

	$\Delta_{MCS}(E)$	$\Delta_{max}(E)$	$\Delta_{\Sigma}(E)$	$\Delta_{Gmax}(E)$
$i_w(K_1, \Delta_a(E))$	1	1	1	1
$i_w(K_2, \Delta_a(E))$	1	1	1	1
$i_w(K_3, \Delta_a(E))$	1	1	1	1
$i_{w,min}(E, \Delta_a(E))$	1	1	1	1
$i_{w,\Sigma}(E, \Delta_a(E))$	3	3	3	3
$i_s(K_1, \Delta_a(E))$	0	0	1	1
$i_s(K_2, \Delta_a(E))$	0	0	0	0
$i_s(K_3, \Delta_a(E))$	0	0	0	0
$i_{s,min}(E, \Delta_a(E))$	0	0	0	0
$i_{s,\Sigma}(E, \Delta_a(E))$	0	0	1	1
$i_p(K_1, \Delta_a(E))$	0.66	0.66	1	1
$i_p(K_2, \Delta_a(E))$	0.66	0.66	0.5	0.5
$i_p(K_3, \Delta_a(E))$	0.33	0.33	0.5	0.5
$i_{p,min}(E, \Delta_a(E))$	0.33	0.33	0.5	0.5
$i_{p,\Sigma}(E, \Delta_a(E))$	1.66	1.66	2	2
$i_d(\ldots, \Delta_a(E))$	same as i_w			
$i_L(\ldots, \Delta_a(E))$	same as i_w			

Table 2. Satisfaction indexes for Example 1.

Using the strong drastic index i_s for Example 1, the results are (in almost all cases) 0, meaning the bases are inconsistent with the merged base. i_p produces a greater degree of granularity in the results which means it is a more discriminative index. In both examples, the new base satisfaction index i_L shows that the Δ_{MCS} merging operator will be maximally satisfied for each belief base K_i as long as K_i is consistent. Likewise, the profile satisfaction indexes $i_{L,min}$ and $i_{L,\Sigma}$ will be maximally satisfied, as long as $\forall K_i \in E$, K_i is consistent. In both examples, the i_L and i_d indexes produce the same results. The reason is: firstly, they are both normalized with the number of variables in the merged base; and secondly, in these examples, the number of inconsistent variables is equal to the minimum distance between models in K_i and the merged base.

In [7] the authors claim for Example 3 that Δ_{Gmax} selects the interpretations chosen by both Δ_{max} and Δ_{Σ}, showing its good behavior, however they do not

	$\Delta_{MCS}(E)$	$\Delta_{max}(E)$	$\Delta_{\Sigma}(E)$	$\Delta_{Gmax}(E)$
$i_w(K_1, \Delta_a(E))$	1	1	1	1
$i_w(K_2, \Delta_a(E))$	1	0	1	0
$i_w(K_3, \Delta_a(E))$	1	0	0	0
$i_{w,min}(E, \Delta_a(E))$	1	0	0	0
$i_{w,\Sigma}(E, \Delta_a(E))$	3	1	2	1
$i_s(K_1, \Delta_a(E))$	0	0	1	0
$i_s(K_2, \Delta_a(E))$	0	0	1	0
$i_s(K_3, \Delta_a(E))$	0	0	0	0
$i_{s,min}(E, \Delta_a(E))$	0	0	0	0
$i_{s,\Sigma}(E, \Delta_a(E))$	0	0	2	0
$i_p(K_1, \Delta_a(E))$	0.5	0.33	1	1
$i_p(K_2, \Delta_a(E))$	0.5	0	0.5	0
$i_p(K_3, \Delta_a(E))$	0.5	0	0	0
$i_{p,min}(E, \Delta_a(E))$	0.5	0	0	0
$i_{p,\Sigma}(E, \Delta_a(E))$	1.5	0.33	1.5	1
$i_d(K_1, \Delta_a(E))$	1	1	1	1
$i_d(K_2, \Delta_a(E))$	1	0.66	1	0.66
$i_d(K_3, \Delta_a(E))$	1	0.66	0.66	0.66
$i_{d,min}(E, \Delta_a(E))$	1	0.66	0.66	0.66
$i_{d,\Sigma}(E, \Delta_a(E))$	3	2.33	2.66	2.33
$i_L(\ldots, \Delta_a(E))$	same as i_d			

Table 3. Satisfaction indexes for Example 3.

provide a formal definition of 'good behavior'. Our proposal, on the other hand, allows us to provide this definition: using i_d and $i_{d,\Sigma}$, we can conclude that Δ_{Gmax} is stronger than Δ_{max} and Δ_{Σ}, moreover, $\Delta_{Gmax}(E)$ is fairer and more satisfactory than $form(\Omega_{max})$ and $form(\Omega_{\Sigma})$ (the 'extra' worlds of $\Delta_{max}(E)$ and $\Delta_{\Sigma}(E)$, respectively). So, we can conclude that the result given by Δ_{Gmax} is better than the results given by Δ_{max} and Δ_{Σ}.

6 Conclusion

We proposed a method for measuring the result of different merging operators. Firstly, we defined a relation of conformity over operators in order to classify the degree to which an operator conforms to six postulates describing the principles that a belief merging operator should satisfy. Next, we introduced the notion of a degree of satisfaction of belief bases. We discovered that some base satisfaction indexes found in the literature satisfy the definition of a degree of satisfaction of belief bases, so we use them to define a profile satisfaction index. Based on the notion of a degree of satisfaction and a profile satisfaction index, we defined two more ordering relations over merging operators: fairness and satisfaction. However, by using these relations the measure of operators does not give intuitive classifications, for example operators such as $\Delta_{\top}$ are well placed, even though the result is neither informative nor useful. So, a fourth relation over operators was introduced, called strength, in order to address this issue. Even while using the four proposed relations, some operators are not fully comparable. This means that we cannot find a best operator for every profile. However the relations do allow us to find the best operators for a given profile.

The proposed method is as follows: first, determine the conformity of an operator, next, if an operator Δ_1 is stronger than an operator Δ_2 for a profile E we can continue, otherwise stop since comparison is not possible. Choose degree of satisfactions SAT and SAT_a in order to compare the operators. Find Ω: the worlds that are included in $\Delta_2(E)$ but not in $\Delta_1(E)$. If $\Delta_1(E)$ is fairer and more satisfactory than Ω in terms of SAT and SAT_a then Δ_1 provides a better result than Δ_2 for the fixed profile E given SAT and SAT_a. While the method is in a preliminary phase, the application on some examples from the literature allows us to formally demonstrate claims such as the 'good behavior' of Δ_{Gmax}.

Our proposed method does not work with integrity constraints however these will be considered in future work. Also, currently we only consider flat belief bases, but we intend to extend this for prioritized bases. In terms of aggregation functions, we analyzed min and Σ for generating the satisfaction index of a profile, however there are other functions available, such as $Gmin$, which could be analyzed. We also intend to propose a profile satisfaction index based on formula-level inconsistency measures.

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