Revision over partial pre-orders: a postulational study

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Abstract. Belief revision is the process that incorporates, in a consistent way, a new piece of information, called input, into a belief base. When both belief bases and inputs are propositional formulas, a set of natural and rational properties, known as AGM postulates, have been proposed to define genuine revision operations. This paper addresses the following important issue : How to revise a partially pre-ordered information (representing initial beliefs) with a new partially pre-ordered information (representing inputs) while preserving AGM postulates? We first provide a particular representation of partial pre-orders (called units) using the concept of closed sets of units. Then we restate AGM postulates in this framework by defining counterparts of the notions of logical entailment and logical consistency. In the second part of the paper, we provide some examples of revision operations that respect our set of postulates. We also prove that our revision methods extend well-known lexicographic revision and natural revision for both cases where the input is either a single propositional formula or a total pre-order.

1 Introduction

The problem of belief revision is a major issue in several Artificial Intelligence applications to manage the dynamics of information systems. Roughly speaking, belief revision results from the effect of inserting new piece of information while preserving some consistency conditions. In the logical setting, a simple form of a belief revision assumes that both initial beliefs, denoted by K, and input information, denoted by μ, are represented by propositional formulas. In this framework, the revision of K by μ consists in producing a new formula denoted by $K * \mu$, where $*$ represents a revision operation. Extensive works have studied and characterized the revision operation $*$ from semantics, syntactic, computational and axiomatics points of views. In particular, Alchourron, Gärdenfors and Makinson [1] proposed an elegant set of rationality postulates [14], known as AGM postulates, that any revision operation $*$ should satisfy. These postulates are mainly based on two important principles: success principle and minimal change principle. The success principle states that the input μ is a sure piece of information and hence should be entailed from $K * \mu$. The minimal change principle states that the revised base $K * \mu$ should be as close as possible to initial beliefs K.

In particular if K and μ are consistent then $K * \mu$ should be simply equivalent to the propositional conjunction of K and μ.

Since AGM proposal, many extensions [10] have been proposed to take into account complex belief and inputs. For instance, in the context of uncertain information, a so-called Jeffrey's rule [12, 7], has been proposed for revising probability distributions. In evidence theory, revision of mass functions are proposed [26, 27]. Similarly, in possibility theory and ordinal conditional functions framework, the so-called transmutation [33] have been proposed. It modifies the ranking or possibility degrees of interpretations so as to give priority to the input information. Various forms of ranking revisions have been suggested in (e.g., [6, 13, 8, 22, 9, 23, 24]).

In the logical setting, belief revision has also been extensively studied. In [8] four postulates have been added to AGM postulates in order to characterize iterated belief revision operators which transform a given ordering on interpretations, in presence of new information, into a new ordering. In [29, 3] a lexicographic strategy, associated with a set of three rationality postulates, have been defined to revise a total pre-order by a new total pre-order. In [5, 4], different strategies have been proposed to revise an epistemic state represented by a partial pre-order on the possible worlds.

This paper deals with a flexible representation of information where both initial beliefs and input are represented by partial pre-orders. Despite its importance in many applications, there are very few works that address revision methods of a partial pre-order by a partial pre-order. In [32], revision of partial orders is studied in a standard expansion and contraction way. But it does not provide concrete revision results because of the use of certain kinds of selection functions. In a short paper [21], a framework that studies revision on partial pre-orders is developed and two main revision operators are proposed. However, there are no postulates addressed in that short paper, neither did the paper discussed the relationship between the revision operators and various revision strategies proposed in the literature.

A natural question addressed in this paper is whether it is possible to reuse AGM postulates while both initial beliefs and inputs are partial pre-orders. The answer is Yes. The idea is not to change rationality postulates, but to modify the representation of beliefs and adapt main logical concepts such as logical entailment and consistency. More precisely, we will follow the representation of partial pre-orders proposed in [21]. A partial pre-oder over a set of symbols is viewed as a closed set of *units*. Each unit represents an individual constraint between symbols (a pair of symbols with an ordering connective). The revision of partial pre-order by another partial pre-order is then viewed as revision of closed set of units by another closed set of units. In this paper, we show that AGM postulates have natural counterparts when initial beliefs and input are represented by sets of units. We also provide the counterpart of success postulates and minimal change principle in our frameworks. The reformulation of AGM postulates is possible once logical entailment is interpreted as set inclusion between sets of units, and logical consistency is interpreted as the absence of cycles between sets of units associated with partial pre-orders of inputs and initial beliefs. Additional postulates are also studied in this paper.

In the second part of the paper, we prove that the two revision operators proposed in [21] satisfy the proposed postulates for partial pre-order revision as well as some rational properties, such as the iteration property. Furthermore, we also prove that our revision operators are extensions of the well-known lexicographic revision [30] and natural belief revision [6].

To summarize, this paper makes the following main contributions:

- We propose a set of rationality postulates that the unit-based revision shall follow.
- We prove some important properties among these revision strategies. We also prove that these revision strategies satisfy certain rationality postulates.
- When reducing to classical belief revision, we prove that our revision strategies extend some existing revision strategies, such as, lexicographic and natural revisions.

The remainder of the paper is organized as follows. We introduce some necessary notations and definitions in Section 2. In Section 3, we discuss the principles a revision rule on partial pre-orders shall satisfy and propose AGM-style postulates for partial pre-orders revision. We then discuss, in Sections 4, two examples of belief revision strategies, proposed in [21], that satisfy AGM-style postulates and analyse their properties. Section 5 shows that the well-known lexicographic and natural revision can be recovered in our frameworks. We then review some related works in Section 6. Section 7 concludes the paper.

2 Notations and Definitions

We use W to denote a finite set of symbols. Let $\preceq$ be a pre-order over W where $w \preceq w'$ means that w is at least as preferred as w'. Two operators, $\prec$ and $\approx$, are defined from $\preceq$ in a usual sense. Note that as $\preceq$ implies $\prec$ or $\approx$ while $\prec$ (or $\approx$) is a simple relation, in this paper, we only focus on simple relations $\prec$ and $\approx$. Each $w \prec w'$ or $w \approx w'$ is called a *unit* for $w \neq w'$. A partial pre-order is represented by a finite set of units denoted by S, we use $Sym(S)$ to denote the set of symbols from W appearing in S.

Definition 1 *A set of units S is closed iff*

- *$w \approx w' \in S$ implies $w' \approx w \in S$;*
- *for any different w_1, w_2, w_3 in W, if $w_1 \ R_1 \ w_2 \in S$ and $w_2 \ R_2 \ w_3 \in S$ and $w_1 \ R_1 \ w_2 \wedge w_2 \ R_2 \ w_3$ implies $w_1 \ R_3 \ w_3$, then $w_1 \ R_3 \ w_3 \in S$, (where R_i is either $\approx$ or $\prec$).*

We can see that a closed set of units corresponds to a partial pre-order in the usual sense (i.e., a transitive, reflexive binary relation). A set S can be extended to a unique minimal closed set based on transitivity and symmetry of $\approx$ and transitivity of $\prec$. We use $Cm(S)$ to denote this unique minimal closed set extended from S.

Example 1 *Let $S = \{w_1 \approx w_2, w_2 \prec w_3\}$, then $Cm(S) = \{w_1 \approx w_2, w_2 \approx w_1, w_2 \prec w_3, w_1 \prec w_3\}$.*

S is closed when it cannot be extended further. This is the counterpart of the deductive closure of a belief base K in classical logics.

Definition 2 *A subset C of S is a cycle if* $C = \{w_1\ R_1\ w_2, w_2\ R_2\ w_3, \cdots, w_n\ R_n\ w_1\}$ *s.t.* $\exists R_i, R_i$ *is* $\prec$ *for* $1 \leq i \leq n$. C *is minimal if there does not exist a cycle* C' *s.t.* $Cm(C') \subset Cm(C)$.

If S has a cycle, then S is said to be inconsistent. Otherwise it is said to be consistent or free of cycles. If S is closed and contains cycles, then all minimal cycles are of the form $\{a \prec b, b \prec a\}$ or $\{a \prec b, b \approx a\}$, i.e., only two units.

Any unit $w\ R\ w'$ is called a *free unit* if $w\ R\ w'$ is not involved in any cycle in S. The concept of free unit is the counterpart of free formula concept in logic-based inconsistency handling [2, 11].

Example 2 *Let* $S = \{w_1 \prec w_2, w_2 \approx w_3, w_3 \approx w_4, w_4 \prec w_1, w_3 \prec w_1\}$*, then* $C_1 = \{w_1 \prec w_2, w_2 \approx w_3, w_3 \prec w_1\}$ *is a minimal cycle whilst* $C_2 = \{w_1 \prec w_2, w_2 \approx w_3, w_3 \approx w_4, w_4 \prec w_1\}$ *is a cycle but not minimal, since the sub sequence* $w_3 \approx w_4, w_4 \prec w_1$ *in* C_2 *can be replaced by* $w_3 \prec w_1$ *and hence forms* C_1.

For any set of units S, we use $[S]$ to count the number of semantically distinct units in S such that $w \approx w'$ and $w' \approx w$ are counted as one instead of two. So for $S = \{w_1 \approx w_2, w_2 \approx w_1, w_3 \prec w_1, w_3 \prec w_2\}$, we have $[S] = 3$.

Without loss of generality, subsequently, if without other specifications, we assume that a set of units S and any new input S_I are both closed and free of cycles. For convenience, we use $\mathcal{S}_{CC}$ to denote the set of all closed and consistent sets of units (free of cycles) w.r.t. a given W and $\{\approx, \prec\}$.

3 Principles and Postulates of unit-based revision

3.1 Motivations

Let $\odot$ be a revision operator which associates a resultant set of units $\hat{S} = S \odot S_I$ with two given sets, one represents the prior state (S) and the other new evidence (S_I). This section provides natural properties, for the unit-based revision operation $\odot$, which restate the AGM postulates [14] in our context. As in standard belief revision (an input and initial are sets of propositional formulas), we also consider the two following principles as fundamental:

Success postulate: It states that information conveyed by the input evidence should be retained after revision. In our context, this also means that an input partial pre-order (or its associated set of units) must be preserved, namely $S_I \subseteq S \odot S_I$. In particular if two possible worlds have the same possibility conveyed by the input, then they should still be equally possible after revision, regardless their ordering in the prior state. This clearly departs from the work reported in [3] (where a tie in the input could be broken by the prior state) for instance. In the degenerate case, where the input is fully specified by a total pre-oder then the result of revision should be simply equal to the input. This situation is similar to the case where in standard AGM postulates where the input is a propositional formula having exactly one model (and hence the result of revision is that formula), or in frameworks of probabilistic revision where applying Jeffrey's rule of conditioning to the situation where the input if specified by a probability distribution simply gives that probability distribution.

Minimal change principle: It states that the prior information should be altered as little as possible while complying with the Success postulate. This means in our context that as few of units (individual binary ordering relations) as possible to be removed from the prior state after revision. Specifying minimal change principle needs to define the concept of conflicts in our framework. Roughly speaking, two sets of units (here associated with input and initial beliefs) are conflicting if the union of their underlying partial pre-orders contains cycles. Some units (from the set of units representing initial beliefs) should be removed to get rid of cycles, and minimal change requires that this set of removed units should be as small as possible.

3.2 AGM-style postulates for unit-based revision

We now rephrase basic rationality postulates when initial epistemic state and input are no longer propositional formulas but sets of units. More precisely, we adapt the well-known AGM postulates (reformulated in [14] (KM)) to obtain a set of revised postulates in which an agent's original beliefs and an input are represented as sets of units. These revised postulates, dubbed UR0-UR6, are as follows:

UR0 $S \odot S_I$ is a closed set of units.
UR1 $Cm(S_I) \subseteq S \odot S_I$.
UR2 If $S_I \cup S$ is consistent,
then $S \odot S_I = Cm(S_I \cup S)$.
UR3 If S_I is consistent, then $S \odot S_I$ is also consistent.
UR4 If $Cm(S_1) = Cm(S_2)$ and $Cm(S_{I1}) = Cm(S_{I2})$,
then $S_1 \odot S_{I1} = S_2 \odot S_{I2}$.
UR5 $S \odot (S_{I1} \cup S_{I2}) \subseteq Cm((S \odot S_{I1}) \cup S_{I2})$.
UR6 If $(S \odot S_{I1}) \cup S_{I2}$ is consistent,
then $Cm((S \odot S_{I1}) \cup S_{I2}) \subseteq S \odot (S_{I1} \cup S_{I2})$.

UR0 simply states that $S \odot S_I$ is closed based on the five inference rules on units (Def. 1).

UR1 formalizes the success postulate. Note that the counterpart of logical entailment here is represented by set inclusion. However it departs from the standard propositional logic definition of entailment where $\phi \models \mu$ means that the set of models of ϕ is included in the set of models of μ. In our context, a set of units S is said to entail another set of units S_1 if $Cm(S_1) \subseteq Cm(S)$.

UR2 indicates that if the prior state and the input are consistent, then the revision result is simply the minimal closure of the disjunction of the prior state and the input. Here we need to point out that consistency here is represented by the absence of cycles. Our definition of consistency (between two sets of units) is stronger than that in logic-based revision (generally defined between two propositional formulae representing the initial beliefs and an input). That is, in belief revision, consistency is required between two formulas (initial state vs. input). In this paper, consistency is required for pre-orders (initial state vs. input) which contain more information than formulas. This is similar to asking for the consistency between two epistemic states not the consistency between their belief sets. Therefore, **UR2** is stronger than its counterpart R2 in logic-based revision [8]

R2: if $\Phi \wedge \mu$ is consistent, then $Bel(\Phi \circ \mu) \equiv Bel(\Phi) \wedge \mu$,
or its weaker version defined when initial epistemic state is represented by a partial pre-order [4, 5]: if $Bel(\Phi) \models \mu$, then $Bel(\Phi \circ \mu) \equiv Bel(\Phi) \wedge \mu$. This is not surprising, since in our framework, all units play an equal role in the belief change, while in logic-based revision, only units that determine belief sets play crucial roles[1].

Also, here we should point out that we do not define Bel in our framework because in general, elements (w_1, w_2) can be more than possible worlds. However, when reducing to belief revision scenarios, Bel cab be defined in the standard way.

UR3 ensures consistency of the revision result given the consistency of the input.

UR4 is a kind of *syntactic irrelevance* postulate, in which we use minimal closure equivalence to replace logical equivalence used in usual syntactic irrelevance postulates.

UR5 and **UR6** depict the conditions that the order of revision and disjunction operations is exchangeable.

In summary, this set of postulates are natural counterparts of KM postulates in our framework.

Remarks: Most of existing approaches modify AGM postulates to cope with new revision procedures (e.g. revising total pre-orders). Our approach brings a fresh perspective to the problem of representing/revising pre-orders in which AGM can be used. Our results may retrospectively appear to be obvious but note that it has not been considered before. Actually, it was not obvious that in order to revise a partial pre-order by another (where a new representation of partial pre-order is needed), one can still use AGM postulates.

3.3 Additional postulates

In [25], a postulate proposed for iterated belief revision on epistemic states is defined as follows.

ER4* $\Phi \circ_E \Psi^{\mathscr{F}} \circ_E \Theta^{\mathscr{F}'} = \Phi \circ_E \Theta^{\mathscr{F}'}$ where partition $\mathscr{F}'$ of W is a refinement of partition $\mathscr{F}$.

In which $\circ_E$ is an epistemic state revision operator, and Φ, $\Psi^{\mathscr{F}}$, and $\Theta^{\mathscr{F}'}$ are all epistemic states which generalize formula-based belief representations.

In our framework **ER4*** is rewritten as follows:

UER4* For any $S, S_I, S'_I \in \mathcal{S}_{CC}$ such that $S_I \subseteq S'_I$, then $S \odot S_I \odot S'_I = S \odot S'_I$.

UER4* states that given a prior set and two new inputs, if the latter input has a finer structure than the former input, then the latter totally shadows the former in iterated revision. This is clearly the counterpart of **ER4*** and represents an important result for iterated revision.

We also propose additional properties inspired from Darwiche and Pearl's iterated belief revision postulates :

[1] This is an essential difference between our framework with the framework in [4]. As an instance of this difference, R2 is questioned in [4] and hence is not valid in the framework of [4] whilst it is valid (after translation) in our framework.

UCR1 If $w_1 \prec w_2 \in Cm(S_I)$, then $w_1 \prec w_2 \in Cm(S \odot S_I)$.
UCR2 If $w_1 \approx w_2 \in Cm(S_I)$, then $w_1 \approx w_2 \in Cm(S \odot S_I)$.
UCR3 If w_1 and w_2 are incomparable in $Cm(S_I)$, then $w_1 \; R \; w_2 \in Cm(S)$ iff $w_1 \, R \, w_2 \in Cm(S \odot S_I)$ (R is $\prec$ or $\approx$).

Here UCR1-3 are inspired from the semantic expression of DP postulates CR1-CR4. UCR3 is its counterpart of CR1-CR2 in our framework, and it can be seen as a counterpart of the relevance criterion [19]. It says that in case two elements are incomparable in the input, then the ordering between these two elements should be the same in both initial state and revised state. This is inspired from CR1 and CR2 where the ordering of models (resp. countermodels) of input is the same in both initial and revised states.

UCR1-2 are inspired from CR3 and CR2, respectively. Also note that UCR1-2 are implied by UR1.

Note that here any ties (e.g., $w_1 \approx w_2$) in the input are preserved while in the original work of [8], they are broken based on the information given by initial beliefs.

4 Examples of Revision Operators for Partial-Preorders

In this section, we give two examples of revision operators, introduced in [21], that satisfy all postulates.

Match Revision The key idea of match revision is to remove any units in S which join at least one minimal cycle in $S \cup S_I$. Therefore, these units are potentially conflicting with S_I.

Definition 3 *(Match Revision Operator, [21]) For any $S, S_I \in \mathcal{S}_{CC}$, let $S' = Cm(S \cup S_I)$ and let $\mathcal{C}$ be the set of all minimal cycles of S', then the match revision operator $\odot_{match}$ is defined as:*
$S \odot_{match} S_I = Cm(S' \setminus (\bigcup_{C \in \mathcal{C}} C \setminus S_I))$.

Example 3 *Let $S = \{w_3 \prec w_2, w_2 \prec w_4, w_4 \prec w_1, w_3 \prec w_4, w_3 \prec w_1, w_2 \prec w_1\}$ and $S_I = \{w_1 \prec w_2, w_4 \prec w_3\}$, then we have six minimal cycles in $Cm(S \cup S_I)$. That is*

$C_1 : w_1 \prec w_2, w_2 \prec w_1, \quad C_2 : w_1 \prec w_4, w_4 \prec w_1,$
$C_3 : w_2 \prec w_4, w_4 \prec w_2, \quad C_4 : w_3 \prec w_2, w_2 \prec w_3,$
$C_5 : w_3 \prec w_4, w_4 \prec w_3, \quad C_6 : w_1 \prec w_3, w_3 \prec w_1.$

Hence we have: $S \odot_{match} S_I = Cm(\{w_1 \prec w_2, w_4 \prec w_3\}) = \{w_1 \prec w_2, w_4 \prec w_3\}$.

However, the match revision operator removes too many units from the prior state S, as we can see from Example 3. In fact, if certain units are removed from S, then there will be no cycles in $S \cup S_I$, hence some other units subsequently could have been retained. That is, there is no need to remove all the conflicting units at once, but one after the other. This idea leads to the following inner and outer revision operators.

Inner Revision The basic idea of inner revision is to insert each unit of S_I one by one into S, and in the meantime, remove any unit in S that are inconsistent with the inserted unit. Of course, the revision result depends on the order in which these units from S_I are inserted to S. Hence, only the units that exist in all revision results for any insertion order should be considered credible for the final, consistent revision result.

For a set of units S, let $\mathsf{PMT}(S)$ denote the set of all permutations of the units in S. For example, if $S = \{w_1 \prec w_3, w_2 \prec w_3\}$, then $\mathsf{PMT}(S) = \{(w_1 \prec w_3, w_2 \prec w_3), (w_2 \prec w_3, w_1 \prec w_3)\}$.

Definition 4 *For any $S, S_I \in \mathcal{S}_{CC}$, let $\overrightarrow{t} = (t_1, \cdots, t_n)$ be a permutation in $\mathsf{PMT}(S_I)$, then the result of sequentially inserting $\overrightarrow{t}$ into S one by one[2], denoted as $S_{\overrightarrow{t}}$, is defined as follows:*

- *Let S_i be the resulted set by sequentially inserting $t_1, \cdots, t_i$ one by one. Let $S' = Cm(S_i \cup \{t_{i+1}\})$ and $\mathcal{C}$ be the set of all minimal cycles of S', then $S_{i+1} = S' \setminus (\bigcup_{C \in \mathcal{C}} C \setminus S_I)$.*
- *$S_{\overrightarrow{t}} = S_n$.*

The inner revision operator is defined as follows.

Definition 5 *(Inner Revision Operator, [21]) For any $S, S_I \in \mathcal{S}_{CC}$, the inner revision operator is defined as:*

$$S \odot_{in} S_I = Cm(\bigcap_{\overrightarrow{t} \in \mathsf{PMT}(S_I)} S_{\overrightarrow{t}}). \tag{1}$$

Example 4 *Let $S = \{w_3 \prec w_2, w_2 \prec w_4, w_4 \prec w_1, w_3 \prec w_4, w_3 \prec w_1, w_2 \prec w_1\}$ and $S_I = \{w_1 \prec w_2, w_4 \prec w_3\}$, then we have $S_{(w_4 \prec w_3, w_1 \prec w_2)} = \{w_1 \prec w_2, w_4 \prec w_3, w_3 \prec w_1, w_4 \prec w_1, w_4 \prec w_2, w_3 \prec w_2\}$ and $S_{(w_1 \prec w_2, w_4 \prec w_3)} = \{w_1 \prec w_2, w_4 \prec w_3, w_3 \prec w_1, w_3 \prec w_2, w_4 \prec w_2, w_4 \prec w_1\}$. Hence $S \odot_{in} S_I = \{w_1 \prec w_2, w_4 \prec w_3, w_3 \prec w_1, w_3 \prec w_2, w_4 \prec w_2, w_4 \prec w_1\}$.*

Examining unit-based postulates For operators $\odot_{in}$ and $\odot_{match}$, we have the following results.

Proposition 1 *The revision operators $\odot_{in}$ and $\odot_{match}$ satisfy UR0-UR6.*

This proposition shows that our revision operators satisfy all the counterparts of AGM-style postulates proposed in Section 3.

Next we show that our revision operations satisfy **UER4*** and iteration properties:

Proposition 2
- *For any $S, S_I, S'_I \in \mathcal{S}_{CC}$ such that $S_I \subseteq S'_I$, then $S \odot_{in} S_I \odot_{in} S'_I = S \odot_{in} S'_I$. We also have $S \odot_{match} S_I \odot_{match} S'_I = S \odot_{match} S'_I$.*
- *The revision operators $\odot_{in}$ and $\odot_{match}$ satisfy UCR1-UCR3.*

To summarize, from the Proposition 2, we can conclude that match and inner revision operator satisfies many of the well-known postulates for belief revision and iterated belief revision, which demonstrates that these strategies are rational and sound.

[2] As two units like $w \approx w'$ and $w' \approx w$ are in fact the same, in this and the next section, this type of units are considered as one unit and be inserted together.

5 Recovering lexicographic and natural belief revision

Now we come to show that the two well-known belief revision strategies (lexicographic and natural revision) can be encoded as stated in our framework.

5.1 Recovering lexicographic revision

We first assume that an input, representing a propositional formula μ (a typical input in the belief revision situation), is described by a *modular order* where:

- each model of μ is preferred to each model of $\neg\mu$,
- models (resp. counter-models) of μ are incomparable.

That is, given μ, the corresponding input representing μ is denoted as $S_I^\mu = \{w \prec w' : \forall w \models \mu, w' \models \neg\mu\}$. Clearly for any consistent μ, $S_I^\mu \in \mathcal{S}_{CC}$.

Then we have the following result.

Proposition 3 *For any $S \in \mathcal{S}_{CC}$ and any consistent μ, we have $S \odot_{in} S_I^\mu = S_I^\mu \cup \{wRw' : w, w' \models \mu \text{ and } wRw' \in S\} \cup \{wRw' : w, w' \models \neg\mu \text{ and } wRw' \in S\}$.*

Proof of Proposition 3: A sketch of the proof can be shown as follows. Let $\hat{S} = S \odot_{in} S_I^\mu$, then it is easy to show the following steps:

- $S_I^\mu \subseteq \hat{S}$.
  The input is reserved in the revision result.
- $\{wRw' : w, w' \models \mu \text{ and } wRw' \in S\} \in \hat{S}$, $\{wRw' : w, w' \models \neg\mu \text{ and } wRw' \in S\} \in \hat{S}$.
  It is easy to see that any unit $wRw' \in S$ such that w, w' are both models of μ or both counter-models of μ is consistent with S_I^μ. So any such unit is in $\hat{S}$.
- $\hat{S}$ only contains units which can be induced from the above two steps. In fact, we can find that $S_I^\mu \cup \{wRw' : w, w' \models \mu \wedge wRw' \in S\} \cup \{wRw' : w, w' \models \neg\mu \wedge wRw' \in S\}$ already forms a total pre-order over W (and hence is complete) and obviously it is consistent. $\square$

That is, given the input S_I^μ representing μ, inner revision operator reduces to a lexicographic revision [30] in the belief revision case. Hence obviously it follows the spirit of AGM postulates [1], Darwiche and Pearl's iterated belief revision postulates [8], and the Recalcitrance postulate [30].

In [3] an extension of lexicographic revision of an epistemic state $\rhd_{initial}$ (viewed as a total pre-order), by an input in the form of another total pre-order, denoted here by $\rhd_{input}$, is defined. The obtained result is a new epistemic state, denoted by $\rhd_{lex}$ (lex for lexicographic ordering), and defined as follows:

- $\forall w_1, w_2 \in W$, if $w_1 \rhd_{input} w_2$ then $w_1 \rhd_{lex} w_2$.
- $\forall w_1, w_2 \in W$, if $w_1 =_{input} w_2$ then $w_1 \rhd_{lex} w_2$ if and only if $w_1 \rhd_{initial} w_2$.

Namely, $\rhd_{lex}$ is obtained by refining $\rhd_{input}$ by means of the initial ordering $\rhd_{initial}$ for breaking ties in $\rhd_{input}$.

To recover this type of revision, it is enough to interpret ties in $\rhd_{input}$ as incomparable relations, namely:

Proposition 4 *Let $\rhd_{initial}$ and $\rhd_{input}$ be two total pre-orders and $\rhd_{lex}$ be the result of refining $\rhd_{input}$ by $\rhd_{initial}$ as defined above. Let $S_{initial}$ be the set of all units in $\rhd_{initial}$, and S_{input} be the set of strict relations in $\rhd_{input}$, namely, $S_{input} = \{\omega_1 \prec \omega_2$ such that $\omega_1 \rhd_{input} \omega_2\}$ (i.e., ties in $\rhd_{input}$ are not included in S_{input}). Then we have : $S_{initial} \odot_{in} S_{input} = Cm(\{\omega_1 \prec \omega_2$ such that $\omega_1 \rhd_{lex} \omega_2\})$.*

The above proposition shows that the lexicographic inference can be recovered when both initial beliefs and input are total pre-ordered. Of course, our approach goes beyond lexicographic belief revision since inputs can be partially pre-ordered.

5.2 Recovering natural belief revision

This section shows that inner revision allows to recover the well-known natural belief revision proposed in [6] and hinted by Spohn [31]

Let $\rhd_{initial}$ be a total pre-order on the set of interpretations representing initial epistemic state. Let μ be a new piece of information. We denote by $\rhd_N$ he result of applying natural belief revision of $\rhd_{initial}$ by ϕ. $\omega =_N \omega'$ denotes that ω and ω' are equally plausible in the result of revision. Natural belief revision of $\rhd_{initial}$ by μ consists in considering the most plausible models of μ in $\rhd_{initial}$ as the most plausible interpretations in $\rhd_N$. Namely, $\rhd_N$ is defined as follows:

- $\forall\omega \in \min(\phi, \rhd_{initial}), \forall\omega' \in \min(\phi, \rhd_{initial}), \omega =_N \omega'$
- $\forall\omega \in \min(\phi, \rhd_{initial}), \forall\omega' \notin \min(\phi, \rhd_{initial}), \omega \rhd_N \omega'$
- $\forall\omega \notin \min(\phi, \rhd_{initial}), \forall\omega' \notin \min(\phi, \rhd_{initial}), \omega \rhd_N \omega'$ iff $\omega \rhd_{initial} \omega'$.

In order to recover natural belief revision we will again apply inner revision. Let us describe the input. We will denote by ϕ as a propositional formula whose models are those of μ which are minimal in $\rhd_{initial}$. The input is described by the following *modular order* $\rhd_{input}$ where :

- each model of ϕ is preferred to each model of $\neg\phi$, namely : $\forall\omega, \omega'$, if $\omega \models \phi$ and $\omega' \models \neg\phi$ then $\omega \rhd_{input} \omega'$
- models of ϕ are equally plausible, namely : $\forall\omega, \omega'$, if $\omega \models \phi$ and $\omega' \models \phi$ then $\omega =_{input} \omega'$
- models of $\neg\phi$ are incomparable.

Then we have :

Proposition 5 *Let $\rhd_{initial}$ be a total pre-order associated with initial beliefs. Let μ be a proposition formula and $\rhd_N$ be the result of revising $\rhd_{initial}$ by μ as defined above. Let $\rhd_{input}$ be a partial pre-order defined above. Let $S_{initial}$ (resp. $S_{initial}, S_N$) be the set of all units in $\rhd_{initial}$ (resp. $\rhd_{input}, \rhd_N$). Then we have : $S_{initial} \odot_{in} S_{input} = S_N$.*

The above proposition shows that the natural belief revision can be recovered using inner revision. And again, our framework goes beyond natural belief revision since initial beliefs can be partially pre-ordered (while it is defined as totally pre-ordered in [6]).

6 Discussions and Related works

In this section, first we briefly review some related works and then we present some discussions on the difference between our approach and other revision approaches.

[15] and [17] in fact focus on revising with conditionals. Our unit $w \prec w'$ can be translated into $(w|w \vee w')$ in the framework proposed in [17]. However, in [17], they only consider initial and input epistemic states as ordinal conditional functions which cannot encode a partial pre-order in general.

In [32], revision of partial orders is studied in a standard expansion and contraction way, in which contraction uses a cut function which can be seen as a selection function, and hence the result is not deterministic, whilst all revision operators proposed in this paper provide deterministic results.

[28] only considers merging of partial pre-orders (which follows a different definition from this paper) instead of revision, so it departs from the work we investigated in this paper.

Furthermore, we already showed that our approach can recover, as a particular case, the lexicographic inference ([29], etc) when both the initial state and the input are either a propositional formula or a total pre-order.

Lastly, note that our revision operations are totally different from Lang's works on preference (e.g. [20]) and Kern-Isberner's revision with conditionals (e.g., [16]).

Remarks: A key question should be answered is: what is the difference between revision in this paper and in other papers?

In existing approaches, when a revision strategy is extended to deal with some complex task, two steps are commonly followed:

- generalizing the concept of "theory" (initial state) and input. For instance, in [25], when the task is to revise an epistemic state, the representation of initial epistemic state and the input was generalized that can recover almost all common uncertainty representations. However, in most works, these representations extend the concept of "propositional formulas" directly.
- extending or modifying AGM postulates, and several of them "get rid" of some postulates. For instance, in [4], postulate R2 is removed and replaced by other postulates. [18] has suggested new postulates to deal with the idea *improvements* and drop some of the AGM postulates.

In our approach, however, we propose revision operators from a different perspective.

- First, we keep all the AGM postulates even if our aim is to generalize the revision process to deal with a very flexible structure which is a partial pre-order.
- Second, we consider very different components of the revision operation. Initial epistemic state is no longer a propositional formula but a set of units. Similarly for the input. With this change, some standard concepts need to be adapted, in particular the concepts of consistency and entailment. We can illustrate this by the following example taken from [4].

  Example 5 *Let $W = \{w_1, w_2, w_3, w_4\}$. Consider the following partial pre-order representing the agent's initial epistemic state Φ: $w_3 \prec_B w_2 \prec_B w_1$ which means*

w_3 is the most preferred, w_1 is the least preferred, and w_4 is incomparable to others.
Assume that the input is propositional formula μ with models w_2 and w_4, denoted as $[\mu] = \{w_2, w_4\}$ (where $[\phi]$ represents the set of models of any propositional formula ϕ). The aim is to revise the partial pre-order with μ.

Let us see how the two approaches behave (from representational point of view and from axiomatic point of view).
In [4], from a representational point of view,
Epistemic state = Partial pre-order Φ
Input = Propositional formula μ
Output = $\Phi \circ \mu$ which is a propositional formula
From an axiomatic point of view, the example shows that R2 postulate (see R2 in Section 3) maybe questionable. That is, in the example, the former approach only keeps the possible world w_4 that is incomparable to others while the latter also keeps the minimal possible world w_2 in the set of comparable possible worlds, which is more reasonable.
Indeed, in this example:

$$[Bel(\Phi)] = \{w : w \in W \wedge \nexists w', w' \prec_B w\} = \{w_3, w_4\},$$

$$[\mu] = \{w_2, w_4\}.$$

So $Bel(\Phi)$ and μ are consistent and hence according to R2, $Bel(\Phi \circ \mu) = \{w_4\}$. This is questionable as a result.
Therefore, in [4], they consider R2 is no longer valid when epistemic states are partial pre-orders and they propose different alternatives for this postulate.
In our approach, from a representational point of view,
Epistemic state = Partial pre-orders represented by a set of units, i.e., $S = Cm(\{w_3 \prec w_2, w_2 \prec w_1\})$
Input = Partial pre-orders represented by a set of units, i.e., $S_I = \{w_2 \prec w_1, w_2 \prec w_3, w_4 \prec w_1, w_4 \prec w_3\})$

From an axiomatic point of view, we keep all AGM (or KM) postulates but coherence and entailment do not have the same meaning. In our approach, S and S_I are not consistent since in S we have $w_3 \prec w_2$ while in S_I we have $w_2 \prec w_3$. Therefore, the result of revision is not $S \cup S_I$.
What can we obtain with our approach in this example? We get $\hat{S} = Cm\{w_3 \prec w_1, w_2 \prec w_3, w_4 \prec w_3\}$ in which the expected result is obtained with minimal models: $\{w_2, w_4\}$.
In fact, in the definition of our revision operation, we do not focus on $Bel(\Phi)$, instead, we focus on *small components* that compose the partial pre-orders.

7 Conclusion

Although logic-based belief revision is fully studied, revision strategies for ordering information have seldom been addressed. In this paper, we investigated the issue of revising a partial pre-order by another partial pre-order. We proposed a set of rationality postulates to regulate this kind of revision. We also proved several revision operators satisfy these postulates as well as some rational properties.

The fact that our revision operators satisfy the counterparts of the AGM style postulates and the iterated revision postulate shows that our revision strategies provide rational and sound approaches to handling revision of partial pre-orders. In addition, when reducing to classical belief revision situation, our revision strategies become the lexicographic revision. This is another indication that our revision strategies have a solid foundation rooted from the classic belief revision field.

For future work, we will study the relationship between our revision framework and revision strategies proposed for preferences.

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