Under consideration for publication in Knowledge and Information
Systems

A General Framework for Measuring Inconsistency Through Minimal Inconsistent Sets

Kedian Mu[1], Weiru Liu[2] and Zhi Jin[3]

[1]School of Mathematical Sciences, Peking University, Beijing 100871, P.R. China
[2]School of Electronics, Electrical Engineering and Computer Science
Queen's University Belfast, BT7 1NN, UK
[3] Key Laboratory of High Confidence Software Technologies, Ministry of Education
School of Electronics Engineering and Computer Science
Peking University, Beijing 100871, P.R.China

Abstract. Hunter and Konieczny explored the relationships between measures of inconsistency for a belief base and the minimal inconsistent subsets of that belief base in several of their papers. In particular, an inconsistency value termed $\mathsf{MIV_C}$, defined from minimal inconsistent subsets, can be considered as a Shapley Inconsistency Value. Moreover, it can be axiomatized completely in terms of five simple axioms. MinInc, one of the five axioms, states that each minimal inconsistent set has the same amount of conflict. However, it conflicts with the intuition illustrated by the lottery paradox, which states that as the size of a minimal inconsistent belief base increases, the degree of inconsistency of that belief base becomes smaller. To address this, we present two kinds of revised inconsistency measures for a belief base from its minimal inconsistent subsets. Each of these measures considers the size of each minimal inconsistent subset as well as the number of minimal inconsistent subsets of a belief base. More specifically, we first present a vectorial measure to capture the inconsistency for a belief base, which is more discriminative than $\mathsf{MIV_C}$. Then we present a family of weighted inconsistency measures based on the vectorial inconsistency measure, which allow us to capture the inconsistency for a belief base in terms of a single numerical value as usual. We also show that each of the two kinds of revised inconsistency measures can be considered as a particular Shapley Inconsistency Value, and can be axiomatically characterized by the corresponding revised axioms presented in this paper.

Keywords: Inconsistency measure; Belief bases; Minimal inconsistent subsets; Shapley value

Received Sep 14, 2009
Revised Mar 16, 2010
Accepted Mar 27, 2010

1. Introduction

The principle of explosion of classical logic states that anything follows from a contradiction. It renders classical logic not appropriate to describe inconsistent information. For example, all inconsistent sets of classical formulas are equally bad under classical logic, but it is commonly believed that some inconsistent sets are worse than others (Knight, 2002). Logic-based inconsistency management therefore is increasingly recognized as a topic in computer science as well as artificial intelligence (Bertossi et al, 2004; Mu et al, 2007; Bagheri et al, 2010; Fan et al , 2009; Cristani et al, 2009; Resconi et al, 2009).

Measuring inconsistency is considered as a crucial part of an effective and systemic management of inconsistency in many applications. For example, in the field of belief revision and updating, measures of inconsistency for a belief base can be used to guide characterization of belief change operators for belief negotiation as well as for belief revision (Hunter et al, 2006). In the field of software requirements engineering, measures of inconsistency in software requirements provide a good basis for making an useful trade-off decision on resolving conflict among stakeholders (Mu et al, 2005; Mu et al, 2008).

Exploring techniques for measuring inconsistency has attracted significant attention in many applications, such as knowledge merging (Qi et al, 2005), ontology management (Ma et al, 2007), software engineering and requirements negotiation (Mu et al, 2005; Mu et al, 2008; Barrangans-Martinez et al, 2008), as well as in artificial intelligence in general, such as (Grant, 1978; Knight, 2002; Knight, 2003; Bertossi et al, 2004; Hunter et al, 2006; Hunter et al, 2008; Grant et al, 2006; Grant and Hunter, 2008; Konieczny et al, 2003). Some recent approaches for measuring inconsistent information have been reviewed in (Hunter et al, 2004).

The overwhelming majority of the current proposals for measuring inconsistency are concentrated on measuring the degree of inconsistency for a whole belief base. In contrast, there are relatively few techniques for identifying the degree of blame/responsibility of each formula for the inconsistency of a belief base (Hunter et al, 2006; Hunter et al, 2008). However, in many applications such as requirements engineering, it is desirable to choose an appropriate action for resolving the inconsistency of a set of formulas by measuring and identifying the blame or responsibility of each formula for the inconsistency of that set.

Shapley Inconsistency Value presented in (Hunter et al, 2006; Hunter et al, 2008) connected the degree of blame of each formula in a belief base and the degree of inconsistency of that whole belief base together by using a cooperative game theory–Shapley value. Informally, given a measure of inconsistency for the whole belief base, we can apportion the blame for the inconsistency in the belief base to the individual formula in a principled way by using the Shapley value. On the other hand, since the minimal inconsistent subsets of a belief base can be considered as the purest form of inconsistency of that belief base, it is natural to develop measures of inconsistency for a belief base from the minimal inconsistent subsets of that belief base (Hunter, 2004; Hunter et al, 2006; Hunter et al, 2008). Actually, Hunter and Konieczny have already explored connections between measures of inconsistency for a belief base and the minimal inconsistent subsets of that belief base to some extent (Hunter et al, 2008). Especially, an inconsistency value $\mathsf{MIV_C}$ defined from the minimal inconsistent subsets is used to articulate the degree of blame of each formula for the inconsistency of the base. Moreover, it has been shown that this value is the Shapley Inconsistency Value of a coalitional game defined by a basic inconsistency measure, in which the number of minimal inconsistent subsets of a belief base is considered as the payoff available to the grand coalition (i.e. the base). Furthermore, they stated in (Hunter et al, 2008) that this particular Shapley Inconsistency Value can be completely axiomatized in terms of five simple axioms, i.e., Distribution, Symmetry, Minimality, Decomposability and MinInc.

The axiom of MinInc states that each minimal inconsistent set has the same amount of conflict. However, as the cardinality of a minimal inconsistent subset increases, the inconsistency becomes more tolerable (Knight, 2002; Hunter et al, 2006; Hunter et al, 2008). That is, the bigger the size of the minimal inconsistent subset, the smaller the degree of inconsistency is. To illustrate this, let us consider the lottery paradox which motivated Knight to propose his approach (Knight, 2002). The lottery paradox presented in (Kyburg, 1961) considered an n-ticket lottery known to be fair and to have exactly one winner. It is rational to accept *for any individual ticket i of the lottery that ticket i will not win*, since the probability of ticket i being the winner cannot exceed a high enough threshold due to the fairness of the lottery. Then $K_n = \{\neg w_1, \cdots, \neg w_n, w_1 \vee \cdots \vee w_n\}$ is a minimal inconsistent belief base about the lottery, where for each i, w_i asserts that ticket i will win the lottery. Intuitively, if there are a sufficiently large number of tickets in the lottery, the belief base K_n is *almost consistent*, whilst K_n is *highly inconsistent* if there are only two or three tickets. Evidently, the axiom of MinInc conflicts with this intuition. Correspondingly, the number of minimal inconsistent subsets cannot be considered as a fine-grained basic measure to capture the inconsistency of a belief base.

To address this, in this paper, we give an analysis of this problem, and then we present, as a solution, a more discriminative vectorial inconsistency value defined from minimal inconsistent subsets of a belief base to capture the degree of the blame of each formula in the inconsistency of that belief base. Following that, we provide a family of weighted inconsistency values based on the vectorial inconsistency value, which allow us to measure inconsistency for a belief base in terms of a single numerical value as usual. Both the vectorial inconsistency value and the weighted inconsistency values consider the size of each minimal inconsistent subsets as well as the number of minimal inconsistent subsets. Moreover, we show that each of the two kinds of revised inconsistency value can be considered as a particular Shapley Inconsistency Value and we provide an axiomatic characterization by using the axioms of Distribution, Symmetry, Minimality, and the axioms of Revised Decomposability and Revised MinInc.

The rest of this paper is organized as follows. In the next section, we recall the proposal presented in (Hunter et al, 2008) for measuring inconsistency through minimal inconsistent sets. In Section 3, we analyze the problem of using the axiom of MinInc to characterize inconsistency measurers, and then we provide an alternative to the inconsistency measures presented in (Hunter et al, 2008) in terms of vectors in Section 4. In Section 5, we present a family of weighted inconsistency values from minimal inconsistent subsets. In Section 6, we compare our revised measures with related work. Finally, we conclude this paper in Section 7.

2. Background

First, we give some mathematical notations which will be used later. We use $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\cdots$ to denote the vectors. The lexicographical ordering relation between any two vectors with the same size is given as follows:

Definition 2.1 (Lexicographical ordering relation). Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be two vectors. Suppose that $\mathbf{u} = (u_1, u_2 \cdots, u_n)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$. Then the lexicographical ordering relation $\preceq$ is defined as $\mathbf{u} \preceq \mathbf{v}$ iff

(1) $\mathbf{u} = \mathbf{v}$, or

(2) there exists $k \leq n$ s. t. $u_k < v_k$ and $u_i = v_i$ for each $i < k$.

Furthermore, $\mathbf{u} \prec \mathbf{v}$ iff $\mathbf{u} \preceq \mathbf{v}$ and $\mathbf{u} \neq \mathbf{v}$.

We can then generalize the lexicographical ordering relation to any two vectors with different sizes. Let $\mathbf{u} = (u_1, u_2, \cdots, u_m) \in \mathbb{R}^m$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n) \in \mathbb{R}^n$. Suppose that $m < n$. A n-size extension of $\mathbf{u}$ is defined as $\mathbf{u}' = (u'_1, u'_2, \cdots, u'_n)$ such that $u'_i = \begin{cases} u_i & 1 \leq i \leq m \\ 0, & i > m \end{cases}$. Informally, we write $\mathbf{u} \preceq \mathbf{v}$ (resp. $\mathbf{v} \preceq \mathbf{u}$) if $\mathbf{u}' \preceq \mathbf{v}$ (resp. $\mathbf{v} \preceq \mathbf{u}'$). For example, $(1, 0, 3, 5) \prec (1, 1, 3)$ since $(1, 0, 3, 5) \prec (1, 1, 3, 0)$. Further, we write $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ instead of $\mathbf{u}' + \mathbf{v}$ and $\mathbf{u}' - \mathbf{v}$, respectively.

We use $A \oplus B$ to denote $A \cup B$ for any two mutually exclusive sets A and B (i.e. $A \cap B = \emptyset$).

Throughout this paper, we will use a finite propositional language. Let $\mathcal{P}$ be a finite set of propositional symbols and $\mathcal{L}$ a propositional language built from $\mathcal{P}$. We use a, b, c, $\cdots$ to denote the propositional variables , and α, β, γ, $\cdots$ to denote the propositional formulas.

2.1. Inconsistency Values Defined From Minimal Inconsistent Subsets

A belief base K is a finite set of propositional formulas. We use $\mathcal{K}_{\mathcal{L}}$ to denote the set of belief bases definable from formulas of the language $\mathcal{L}$. A belief base K is inconsistent if there is a formula α such that $K \vdash \alpha$ and $K \vdash \neg\alpha$. We abbreviate $\alpha \wedge \neg\alpha$ as $\perp$ if there is no confusion. Then an inconsistent knowledge base K is denoted by $K \vdash \perp$. Moreover, an inconsistent belief base K is called a *minimal inconsistent set* (or *minimal inconsistent belief base*) if none of its proper subsets is inconsistent. If $K' \subseteq K$ and K' is a minimal inconsistent set, then we call K' a *minimal inconsistent subset* of K.

Let $\mathsf{MI}(K)$ be a set of the minimal inconsistent subsets of K, then

$$\mathsf{MI}(K) = \{K' \subseteq K | K' \vdash \perp \text{ and } \forall K'' \subset K', K'' \nvdash \perp\}.$$

The minimal inconsistent subsets can be considered as the purest form of inconsistency for syntax sensitive conflicts resolution, since one has just to remove one formula from each minimal inconsistent subset in such cases (Reiter, 1987). In contrast, we call a formula of K a *free formula* of K if this formula does not belong to any minimal inconsistent subset of K. That is, the free formulas of K are not involved in the inconsistency of K.

Hunter and Konieczny have argued that it is natural to define measures of inconsistency of each formula of a belief base using only minimal inconsistent subsets of that base (Hunter et al, 2008). This motivates the definition of *MinInc Inconsistency Value* in (Hunter et al, 2008) directly.

Definition 2.2 (MinInc Inconsistency Value). A MinInc Inconsistency Value (MIV) is a function MIV: $\mathcal{K}_{\mathcal{L}} \times \mathcal{L} \longrightarrow \mathbb{R}$ such that $\mathsf{MIV}(K, \alpha) = f(\alpha, \mathsf{MI}(K))$ where f is a function of α and $\mathsf{MI}(K)$.

A family of instances of *the MinInc Inconsistency Value* were also presented in (Hunter et al, 2008). Especially, $\mathsf{MIV_C}$, one of the simplest types of MIV, is considered as an appealing and informative measure of inconsistency, since it considers the number of minimal inconsistent subset of a formula belongs to as well as their cardinalities.

Definition 2.3. $\mathsf{MIV_C}$ is defined as follows:

$$\mathsf{MIV_C}(K, \alpha) = \sum_{M \in \mathsf{MI}(K) s.t. \alpha \in M} \frac{1}{|M|}.$$

Example 2.1. Let $K_1 = \{a, \neg a, \neg a \wedge b, c \wedge \neg c, d\}$. Then the minimal inconsistent subsets of K_1 are $\mathsf{MI}(K_1) = \{\{c \wedge \neg c\}, \{a, \neg a\}, \{a, \neg a \wedge b\}\}$ and the $\mathsf{MIV_C}$ value is given as follows:

$\mathsf{MIV_C}(K_1, a) = \frac{1}{2} + \frac{1}{2} = 1$ $\quad$ $\mathsf{MIV_C}(K_1, \neg a) = \frac{1}{2}$
$\mathsf{MIV_C}(K_1, \neg a \wedge b) = \frac{1}{2}$ $\quad$ $\mathsf{MIV_C}(K_1, c \wedge \neg c) = 1$
$\mathsf{MIV_C}(K_1, d) = 0$

It has been shown in (Hunter et al, 2008) that $\mathsf{MIV_C}$ value is a particular Shapley Inconsistency Value presented in (Hunter et al, 2006).

2.2. Shapley Inconsistency Value

The Shapley Inconsistency Value presented in (Hunter et al, 2006) seeks to define a measure for the degree of blame of each formula in inconsistency from a measure of inconsistency on belief bases, by using a cooperative game theory-the Shapley Value.

We first recall the definitions of the Shapley Value (Aumann et al, 2002) and the Shapley Inconsistency Value (Hunter et al, 2008), respectively .

Definition 2.4. Let $N = \{1, 2, \cdots, n\}$ be a set of n players. A game in coalitional form is given by a function $v : 2^N \longrightarrow \mathbb{R}$, with $v(\emptyset) = 0$.

A coalition C is just a subset of N. The function $v(C)$ gives the payoff which can be achieved by each coalition in the game v when all its members of C act together as a unit.

Definition 2.5. A value is a function that assigns to each game v a vector of payoff $S(v) = (S_1, S_2, \cdots, S_n)$ in $\mathbb{R}^n$, where S_i is the payoff for player i.

The notation of value is referred to as the payoff that can be expected by each player i for the game v.

Definition 2.6. Let $i \in N$ be a player, and n be the number of players. The

Shapley value of player i in the game v is defined as

$$S_i(v) = \sum_{C \subseteq N} \frac{(c-1)!(n-c)!}{n!}(v(C) - v(C \setminus \{i\}))$$

where c is the cardinality of C.

The Shapley value is the first value that gives the expected payoff of each player for any game. Moreover, Shapley's amazing result states that the Shapley value can be characterized uniquely by the following four simple and intuitive axioms (Shapley, 1953):

Proposition 2.1. The Shapley value is the only value that satisfies all of Efficiency, Symmetry, Dummy and Additively.

–**Efficiency**: $\sum_{i \in N} S_i(v) = v(N)$.

–**Symmetry**: If i and j are such that for all C s.t. $i, j \notin C$, $v(C \cup \{i\}) = v(C \cup \{j\})$, then $S_i(v) = S_j(v)$.

–**Dummy**: If i is such that $\forall C$, $v(C \cup \{i\}) = v(C)$, then $S_i(v) = 0$.

–**Additivity**: $S_i(v + w) = S_i(v) + S_i(w)$.

Note that Efficiency states that players precisely distribute among themselves the payoff available to the grand coalition. Symmetry states symmetric players to be paid equal shares. Dummy states that a player whose marginal contribution with respect to any coalition is null, then zero payoffs is assigned to this player. Additivity states that the value is an additive operator on the space of all games.

The Shapley Inconsistency Values combine the Shapley value and the measure of inconsistency for belief bases to identify the blame of each formula in inconsistency. Informally, we first consider an inconsistency measure that allows us to evaluate the inconsistency of a belief base as a game in coalitional form, then compute the corresponding Shapley Value and consider it as the inconsistency value of each formula (Hunter et al, 2006; Hunter et al, 2008).

Hunter and Konieczny in (Hunter et al, 2006; Hunter et al, 2008) provided the following properties to characterize the underlying inconsistency measure.

Definition 2.7. An inconsistency measure I is called a basic inconsistency measure if it satisfies the following properties, $\forall K, K' \in \mathcal{K}_{\mathcal{L}}$, $\forall \alpha, \beta \in \mathcal{L}$:

–**Consistency**: $I(K) = 0$ iff K is consistent.

–**Monotony**: $I(K \cup K') \geq I(K)$.

–**Free Formula Independence**: If α is a free formula of $K \cup \{\alpha\}$, then $I(K \cup \{\alpha\}) = I(K)$.

–**Dominance**: If $\alpha \vdash \beta$ and $\alpha \nvdash \perp$, then $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$.

The Consistency property requires that a desirable inconsistency measure assigns null to a consistency base. The Monotony property states that as a belief base expands, the amount of inconsistency cannot decrease. The Free Formula Independence property requires that adding or deleting a free formula cannot change the amount of inconsistency of the base. As explained in (Hunter et al, 2006), the Dominance property states that logically stronger formulae bring (potentially) more conflicts.

Especially, a basic inconsistency measure, termed the *MI inconsistency measure*, is defined from the minimal inconsistent subsets of a belief base in (Hunter et al, 2008).

Definition 2.8. The *MI inconsistency measure* is defined as the number of minimal inconsistent sets of K, i.e.:

$$\mathsf{I}_{\mathsf{MI}}(K) = |\mathsf{MI}(K)|.$$

Proposition 2.2. The MI inconsistency measure I_{MI} is a basic inconsistency measure.

The Shapley Inconsistency Value presented in (Hunter et al, 2006; Hunter et al, 2008) was aimed to distribute a basic inconsistency measurev (a numerical value) for a belief base among formulas belonging to that belief base by using a coalitional game theory model, i.e., the Shapley Value. The proportion of the basic inconsistency measure distributed to an individual formula of that belief base is considered as a measurement of the responsibility of that formula for the inconsistency of that base.

Definition 2.9 (Shapley Inconsistency Value). (Hunter et al, 2006; Hunter et al, 2008) Let I be a basic inconsistency measure. The corresponding Shapley Inconsistency Value (SIV), denoted S^I, is defined as the Shapley value of the coalitional game defined by the function I, i.e. let $\alpha \in K$:

$$S^I_\alpha(K) = \sum_{C \subseteq N} \frac{(c-1)!(n-c)!}{n!}(I(C) - I(C \setminus \{\alpha\}))$$

where n is the cardinality of K and c is the cardinality of C.

The result shown in (Hunter et al, 2008) is that the MI Shapley Inconsistency Value $S^{\mathsf{I}_{\mathsf{MI}}}$ is exactly the value of $\mathsf{MIV}_{\mathsf{C}}$, i.e.,

Proposition 2.3.

$$S^{\mathsf{I}_{\mathsf{MI}}}_\alpha(K) = \mathsf{MIV}_{\mathsf{C}}(K, \alpha).$$

Moreover, Hunter and Konieczny stated in (Hunter et al, 2008) that this value can be completely axiomatized in terms of five simple axioms, i.e., an inconsistency value satisfies

- **Distribution:** $\sum_{\alpha \in K} S^I_\alpha(K) = I(K)$.
- **Symmetry:** If $\exists \alpha, \beta \in K$ s.t. for all $K' \subseteq K$ s.t. $\alpha, \beta \notin K'$, $I(K' \cup \{\alpha\}) = I(K' \cup \{\beta\})$, then $S^I_\alpha(K) = S^I_\beta(K)$.
- **Minimality:** If α is a free formula of K, then $S^I_\alpha(K) = 0$.
- **Decomposability:** If $\mathsf{MI}(K \cup K') = \mathsf{MI}(K) \oplus \mathsf{MI}(K')$, then $S^I_\alpha(K \cup K') = S^I_\alpha(K) + S^I_\alpha(K')$.
- **MinInc:** If $M \in \mathsf{MI}(K)$, then $I(M) = 1$.

if and only if it is the MI Shapley Inconsistency Value $S^{\mathsf{I}_{\mathsf{MI}}}_\alpha$.

This result was listed as the Proposition 5 in (Hunter et al, 2008). However, there is a minor incorrectness about the proof of this proposition. In the proof of the proposition presented in (Hunter et al, 2008), the role of Decomposability axiom, combined with Minimality axiom, is to deduce the following property:

$$S^I_\alpha(K) = \sum_{M \in \mathsf{MI}(K)} S^I_\alpha(M).$$

Suppose that $\mathsf{MI}(K) = \{M_1, \cdots, M_n\}$. Roughly speaking, if there exists a sequence $M_1, \cdots, M_n$ s.t.

$$\mathsf{MI}(M_1 \cup \ldots \cup M_i \cup M_{i+1}) = \mathsf{MI}(M_1 \cup \ldots \cup M_i) \oplus \mathsf{MI}(M_{i+1})$$

holds for every i $(1 \leq i < n)$, then we can deduce the property stated above by applying Minimality axiom and successive application of Decomposability axiom. However, we cannot find such a sequence of minimal inconsistent subsets for some belief bases. As a counterexample, consider $K = \{a, \neg a \wedge b, \neg a \wedge \neg b\}$. Then $\mathsf{MI}(K) = \{M_1, M_2, M_3\}$, where

$$M_1 = \{a, \neg a \wedge b\}, \quad M_2 = \{a, \neg a \wedge \neg b\}, \quad M_3 = \{\neg a \wedge b, \neg a \wedge \neg b\}.$$

Evidently,

$$\mathsf{MI}(M_i \cup M_j) = \mathsf{MI}(K) \neq \mathsf{MI}(M_i) \oplus \mathsf{MI}(M_j)$$

for any $i \neq j$, $1 \leq i, j \leq 3$. To address this, we propose the following axiom, called Revised Decomposability, to replace the axiom of Decomposability to characterize $S^{\mathsf{I}_{\mathsf{MI}}}_\alpha$ in the proof presented in (Hunter et al, 2008):

– **Revised Decomposability**: $S^I_\alpha(K) = \sum_{M \in \mathsf{MI}(K)} S^I_\alpha(M)$.

Then we can get the following revised proposition about the characterization of $S^{\mathsf{I}_{\mathsf{MI}}}_\alpha$ in this paper:

Proposition 2.4. A Shapley Inconsistency Value satisfies Distribution, Symmetry, Minimality, Revised Decomposability, and MinInc if and only if it is the MI Shapley Inconsistency Value $S^{\mathsf{I}_{\mathsf{MI}}}_\alpha$.

3. The Problem of MinInc Axiom

Recall the axiom of MinInc used to characterize the $S^{\mathsf{I}_{\mathsf{MI}}}$ (or the value $\mathsf{MIV_C}$), it states that each minimal inconsistent subset brings the same amount of conflict, i.e.,

– If $M \in \mathsf{MI}(K)$, then $I(M) = 1$.

Under this assumption, the amount of inconsistency of each minimal inconsistent subset of K is considered as a unit of inconsistency in K. Then the MI inconsistency measure $\mathsf{I_{MI}}(K)$ can be viewed as the total amount of the inconsistency in K. Correspondingly, $\mathsf{MIV_C}(K, \alpha)$ measures the total blame of α for the inconsistency of K.

However, it is not the case that any two minimal inconsistent subsets have the same amount of inconsistency. As illustrated by the lottery paradox, the amount of inconsistency of a minimal inconsistent subset becomes smaller when the size of the minimal inconsistent subset increases. Increasingly, some proposals for evaluating the inconsistency for a belief base seek to accord with this intuition. For example, the notation of *maximal η-consistency* presented by

Knight (Knight, 2002) supports the intuition illustrated by the lottery paradox directly.

We first recall the definition of probability function in (Paris, 1994; Paris et al, 1998), which is considered as a useful tool to formulate the idea of η-consistency (Knight, 2002).

Definition 3.1. A probability function on $\mathcal{L}$ is a function $P : \mathcal{P} \longrightarrow [0, 1]$ s.t.:

–if $\models \alpha$, then $P(\alpha) = 1$,
–if $\models \neg(\alpha \wedge \beta)$, then $P(\alpha \vee \beta) = P(\alpha) + P(\beta)$.

Using the concept of probability function, Knight defined the η-consistency for measuring the inconsistency of a belief base as follows (Knight, 2002):

Definition 3.2. Let K be a belief base.

–K is η-consistent ($0 \leq \eta \leq 1$) if there is a probability function P such that $P(\alpha) \geq \eta$ for all $\alpha \in K$.
–K is maximally η-consistent if η is maximal (i.e., if $\gamma > \eta$ then K is not γ-consistent).

A lower bound of η for a belief base K is also given by the following proposition:

Proposition 3.1 (Corollary 4.12 in (Knight, 2002)). If K is finite and inconsistent but contains no contradictions and $K' \subseteq K$ is a smallest minimal inconsistent subset of K, then K is $\frac{|K'|-1}{|K|}$-consistent.

Evidently, maximal 1-consistency corresponds to complete consistency. And maximal 0-consistency corresponds to the explicit presence of a contradict, i.e., the explicit presence of a contradictory formula in a belief base. Especially, the level of consistency of minimal inconsistent sets of formulas is characterized by the following proposition:

Proposition 3.2 (Theorem 3.5 in (Knight, 2002)). If $K' \in \mathsf{MI}(K)$, then K' is maximally $(1 - \frac{1}{|K'|})$-consistent.

This proposition shows that the maximum level of consistency of a minimal inconsistent set is directly related to its size. In other words, the smaller the size of a minimal inconsistent set, the bigger inconsistency is. The lottery paradox mentioned earlier illustrated the intuition of the maximal η-consistency (Knight, 2002; Hunter et al, 2008).

Example 3.1 (The Lottery Paradox). The n-ticket lottery paradox can be represented by a minimal inconsistent set $K_n = \{\neg w_1, \neg w_2, \cdots, \neg w_n, w_1 \vee w_2 \vee \cdots \vee w_n\}$.

Then K_n is maximally $\frac{n}{n+1}$- consistent. As $n \to \infty$, $\frac{n}{n+1} \to 1$. This explained why K_n is highly inconsistent if there are three or two tickets in the lottery, however, K_n is intuitively (nearly) consistent if there are millions of tickets.

In contrast, the inconsistency value $\mathsf{I}_{\mathsf{MI}}(K_n)$ cannot be used to explain why K_n becomes less inconsistent as the size of K_n increases, since $\mathsf{I}_{\mathsf{MI}}(K_n) = 1$ for every n.

Although the impact of the size of the minimal inconsistent subset on the evaluation of inconsistency has been mentioned in (Hunter et al, 2008), it was not addressed properly in the inconsistency values defined from minimal inconsistent subsets. In detail, given a minimal inconsistent set M,

$$\forall \alpha \in M, \quad \mathsf{MIV_C}(M, \alpha) = \frac{1}{|M|},$$

then as the size of M increases, $\mathsf{MIV_C}(M, \alpha)$ decreases. Compared to the other MIV values defined in (Hunter et al, 2008) such as $\mathsf{MIV}_\sharp(K, \alpha) = |\{M \in \mathsf{MI}(\mathsf{K}) | \alpha \in M\}|$ (in this case $\mathsf{MIV}_\sharp(M, \alpha) = 1$), $\mathsf{MIV_C}(M, \alpha)$ embodies the impact of the size of the minimal inconsistent subset on the evaluation of inconsistency of each formula partially. But consider the sum of the blames of each formula for the inconsistency of M, i.e.,

$$\sum_{\alpha \in M} \mathsf{MIV_C}(M, \alpha) = 1 = \mathsf{I_{MI}}(M).$$

It signifies that all the minimal inconsistent sets have the same total amount of inconsistency. As illustrated by the example of lottery paradox, this conflicts with the intuition that the inconsistency of a minimal inconsistent subset shall decrease as the size of the minimal inconsistent subset increases.

To address this problem, in next two sections, we will provide two kinds of revised inconsistency measures from minimal inconsistent subsets.

4. Vectorial Inconsistency Measures

4.1. MI Inconsistency Vectorial Measure

How to characterize a basic inconsistency measurement has been addressed in (Hunter et al, 2006; Hunter et al, 2008). However, none of these four properties formulates inconsistency in terms of minimal inconsistent sets explicitly. That is, these properties are too general to characterize an inconsistency measure defined from minimal inconsistent subsets of a belief base. Therefore, there is a pressing need for specific properties for characterizing the inconsistency measures defined from minimal inconsistent subsets. To be more intuitive and discriminative, a non-negative (vectorial) inconsistency measurement defined from minimal inconsistent subsets, denoted $\mathbf{I}$, should satisfy the following constraints:

(D1) **Consistency**: If $\mathsf{MI}(K) = \emptyset$, then $\mathbf{I}(K) = \mathbf{0}$.
(D2) **Monotony w.r.t. MI**: $\mathbf{I}(K_1) \preceq \mathbf{I}(K_2)$ if $\mathsf{MI}(K_1) \subseteq \mathsf{MI}(K_2)$.
(D3) **Attenuation**: For any two minimal inconsistent sets K_1 and K_2, $\mathbf{I}(K_1) \prec \mathbf{I}(K_2)$ if $|K_1| > |K_2|$.
(D4) **Equal Conflict**: For any two minimal inconsistent sets K_1 and K_2, $\mathbf{I}(K_1) = \mathbf{I}(K_2)$ if $|K_1| = |K_2|$.
(D5) **Separability**: $\mathbf{I}(K_1 \cup K_2) = \mathbf{I}(K_1) + \mathbf{I}(K_2)$ if $\mathsf{MI}(K_1 \cup K_2) = \mathsf{MI}(K_1) \oplus \mathsf{MI}(K_2)$.

D1 describes the property of Consistency in terms of the minimal inconsistent subsets of a belief base. D2 states that the amount of inconsistency in a belief base increases as the number of its minimal inconsistent subsets increases. D3 requires that the amount of inconsistency in a minimal inconsistent set decreases when the size of a minimal inconsistent set increases, i.e., the bigger the size of a minimal inconsistent set, the smaller the amount of inconsistency it has. D4 states that minimal inconsistent sets with the same cardinality should have the same amount of inconsistency. D5 states that for any two belief bases K_1 and K_2, the amount of inconsistency in $K_1 \cup K_2$ is equal to the sum of the amounts of inconsistency in K_1 and K_2 if $\mathsf{MI}(K_1 \cup K_2) = \mathsf{MI}(K_1) \oplus \mathsf{MI}(K_2)$.

Note that there is no significant difference between D1 and the property of Consistency presented in (Hunter et al, 2006; Hunter et al, 2008). In contrast, D2 may be viewed as a more specific property to characterize the measures defined from minimal inconsistent subsets than the corresponding property of Monotony presented in (Hunter et al, 2006; Hunter et al, 2008). Evidently, if an inconsistency measure satisfies D2, then it must satisfy the property of Dominance as well as the property of Monotony. Moreover, the property of Free Formula Independence can be derived from the combination of D1 and D5.

D3 and D4 are characteristics of the measures defined from minimal inconsistent subsets. In particular, D3 accords with the intuition illustrated by the lottery paradox. These properties make fine-grained inspection of the set of minimal inconsistent subsets of a belief base more necessary. Then we give a partition of minimal inconsistent subsets as follows.

Definition 4.1 (k-size Minimal Inconsistent Subsets). Let $\mathsf{MI}(K)$ be the set of the minimal inconsistent subsets of K and B the size of the biggest minimal inconsistent subsets of K, then for each k ($1 \leq k \leq B$), we define $\mathsf{MI}^{(k)}(K)$ as the set of k-size minimal inconsistent subsets of K, i.e.,

$$\mathsf{MI}^{(k)}(K) = \{\Gamma \in \mathsf{MI}(K) | |\Gamma| = k\}.$$

Evidently, $\mathsf{MI}^{(k_1)}(K) \cap \mathsf{MI}^{(k_2)}(K) = \emptyset$ for any $k_1 \neq k_2$, and $\mathsf{MI}^{(1)}(K) \oplus \cdots \oplus \mathsf{MI}^{(\mathsf{B})}(K) = \mathsf{MI}(K)$. Then we call the B-tuple

$$\langle \mathsf{MI}^{(1)}(K), \cdots, \mathsf{MI}^{(B)}(K) \rangle$$

a partition of the minimal inconsistent subsets. It provides a more fine-grained picture of $\mathsf{MI}(K)$.

Definition 4.2 (MI Cardinality Vector). The MI cardinality vector for a belief base K, denoted $\mathbf{c}(K)$, is defined as

$$\mathbf{c}(K) = (|\mathsf{MI}^{(1)}(K)|, \cdots, |\mathsf{MI}^{(B)}(K)|),$$

where $\langle \mathsf{MI}^{(1)}(K), \cdots, \mathsf{MI}^{(B)}(K) \rangle$ is the partition of the minimal inconsistent subsets of K.

In contrast to $|\mathsf{MI}(K)|$, the element in k-th location of $\mathbf{c}(K)$ gives the number of the k-size minimal inconsistent subsets of K. Then $\mathbf{c}(K)$ considers the size of each minimal inconsistent subset as well as the number of minimal inconsistent subsets with each size. Intuitively, it provides a finer-grained description for the minimal inconsistent subsets of a belief base.

Example 4.1. Consider $K_2 = \{a, a \to b, \neg b, \neg a, c \wedge \neg c\}$. Then $\mathsf{MI}(K_2) = \{\{c \wedge \neg c\}, \{a, \neg a\}, \{a, a \to b, \neg b\}\}$ and $B = 3$. Moreover,

$$\mathsf{MI}^{(1)}(K_2) = \{\{c \wedge \neg c\}\}, \quad \mathsf{MI}^{(2)}(K_2) = \{\{a, \neg a\}\}, \quad \mathsf{MI}^{(3)}(K_2) = \{\{a, a \to b, \neg b\}\}.$$

Therefore $\mathbf{c}(K) = (1, 1, 1)$.

The MI cardinality vector of K can be used to define an inconsistency measure from the minimal inconsistent subsets of K. Then as a revised MI inconsistency measure, MI inconsistency vectorial measure is defined as follows:

Definition 4.3 (MI Inconsistency Vectorial Measure). The MI inconsistency vectorial measure for a belief base K, denoted $\mathbf{I}_{\mathsf{R}}(K)$, is defined as

$$\mathbf{I}_{\mathsf{R}}(K) = \mathbf{c}(K),$$

where $\mathbf{c}(K)$ is the MI cardinality vector of K.

Example 4.2. Consider $K_2 = \{a, a \to b, \neg b, \neg a, c \wedge \neg c\}$ again. Then $\mathbf{I}_{\mathsf{R}}$ values for some subsets of K is given as follows:

$$\begin{array}{ll} \mathbf{I}_{\mathsf{R}}(K_2) = (1,1,1), & \mathbf{I}_{\mathsf{R}}(\{c \wedge \neg c\}) = (1,0,0), \\ \mathbf{I}_{\mathsf{R}}(\{a, \neg a\}) = (0,1,0), & \mathbf{I}_{\mathsf{R}}(\{a, a \to b, \neg b\}) = (0,0,1), \\ \mathbf{I}_{\mathsf{R}}(\{a, c \wedge \neg c\}) = (1,0,0), & \mathbf{I}_{\mathsf{R}}(\{a, \neg b\}) = (0,0,0). \end{array}$$

Proposition 4.1. The MI inconsistency vectorial measure $\mathbf{I}_{\mathsf{R}}$ satisfies (D1)-(D5), i.e.,

(D1) **Consistency**: $\mathbf{I}_{\mathsf{R}}(K) = \mathbf{0}$ iff $\mathsf{MI}(K) = \emptyset$.
(D2) **Monotony w.r.t. MI**: $\mathbf{I}_{\mathsf{R}}(K_1) \preceq \mathbf{I}_{\mathsf{R}}(K_2)$ if $\mathsf{MI}(K_1) \subseteq \mathsf{MI}(K_2)$.
(D3) **Attenuation**: For any two minimal inconsistent sets K_1 and K_2, $\mathbf{I}_{\mathsf{R}}(K_1) \prec \mathbf{I}_{\mathsf{R}}(K_2)$ if $|K_1| > |K_2|$.
(D4) **Equal Conflict**: For any two minimal inconsistent sets K_1 and K_2, $\mathbf{I}_{\mathsf{R}}(K_1) = \mathbf{I}_{\mathsf{R}}(K_2)$ if $|K_1| = |K_2|$.
(D5) **Separability**: $\mathbf{I}_{\mathsf{R}}(K_1 \cup K_2) = \mathbf{I}_{\mathsf{R}}(K_1) + \mathbf{I}_{\mathsf{R}}(K_2)$ if $\mathsf{MI}(K_1 \cup K_2) = \mathsf{MI}(K_1) \oplus \mathsf{MI}(K_2)$.

This proposition shows that the MI inconsistency vectorial measure $\mathbf{I}_{\mathsf{R}}$ is an intuitive inconsistency measure. Moreover, we call a vectorial inconsistency measure *a basic vectorial inconsistency measure* if it satisfies (D1)-(D5).

Note that the MI inconsistency vectorial measure $\mathbf{I}_{\mathsf{R}}$ is linearly homogeneous, i.e.,

$$\mathbf{I}_{\mathsf{R}}(K_2) = t\mathbf{I}_{\mathsf{R}}(K_1) \text{ if } \mathbf{c}(K_2) = t\mathbf{c}(K_1).$$

It signifies that the inconsistency increases in proportion to the MI cardinality vector.

Evidently, $\mathsf{I}_{\mathsf{MI}}(K) = |\mathsf{MI}(K)| = \sum_{i=1}^{B} |\mathsf{MI}^{(i)}(K)|$. The following proposition shows that the MI inconsistency vectorial measure $\mathbf{I}_{\mathsf{R}}$ is *more discriminative* than the MI inconsistency measure I_{MI}.

Proposition 4.2. Let I_{MI} and $\mathbf{I}_{\mathsf{R}}$ be the MI inconsistency measure and the MI inconsistency vectorial measure, respectively. Then $\forall K, K' \in \mathcal{K}_{\mathcal{L}}$,

$$\mathbf{I}_{\mathsf{R}}(K) = \mathbf{I}_{\mathsf{R}}(K') \implies \mathsf{I}_{\mathsf{MI}}(K) = \mathsf{I}_{\mathsf{MI}}(K').$$

But the converse does not hold.

Example 4.3 (A Counterexample for the Converse). Consider $K_3 = \{a, a \to b, \neg b, \neg a\}$ and $K_4 = \{a, a \to b, \neg b, b \wedge \neg b\}$. Then

$$\mathsf{MI}(K_3) = \{\{a,\ \neg a\}, \{a, a \to b, \neg b\}\},$$
$$\mathsf{MI}(K_4) = \{\{b \wedge \neg b\}, \{a, a \to b, \neg b\}\}.$$

So,

$$\begin{array}{ll} \mathsf{I}_{\mathsf{MI}}(K_3) = 2, & \mathsf{I}_{\mathsf{MI}}(K_4) = 2; \\ \mathbf{I}_{\mathsf{R}}(K_3) = (0,1,1), & \mathbf{I}_{\mathsf{R}}(K_4) = (1,0,1). \end{array}$$

Evidently,

$$\mathsf{I}_{\mathsf{MI}}(K_3) = \mathsf{I}_{\mathsf{MI}}(K_4).$$

But

$$\mathbf{I}_{\mathsf{R}}(K_3) \prec \mathbf{I}_{\mathsf{R}}(K_4).$$

Note that $\{a, a \to b, \neg b\}$ is a common minimal inconsistent subsets of K_3 and K_4. Intuitively, the inconsistency level of $\{b \wedge \neg b\}$ is higher than that of $\{a, \neg a\}$. Actually, $\{a, \neg a\}$ is maximally $\frac{1}{2}$-consistent, in contrast, $\{b \wedge \neg b\}$ is maximally 0-consistent. However, in such case, I_{MI} does not consider the distinction between the inconsistency level of $\{a, \neg a\}$ and that of $\{b \wedge \neg b\}$. Therefore, we can not distinguish K_3 from K_4 using I_{MI}. In contrast, $\mathbf{I}_{\mathsf{R}}$ takes the difference between $\{a, \neg a\}$ and $\{b \wedge \neg b\}$ into account, and then can be used to distinguish K_3 from K_4.

Recall the lower bounds of maximal η-consistency for a belief base given by **Proposition** 3.1, if K contains a contradiction (i.e., 1-size minimal inconsistent subset), then K is maximally 0-consistent, whilst K is $\frac{S-1}{|K|}$ consistent if K contains no contradiction, where S is the smallest size of minimal inconsistent subsets of K. It signifies that a 1-size minimal inconsistent subset cannot be replaced by a number of n-size minimal inconsistent subsets in measuring the degree of inconsistency in K. However, for the MI inconsistency vectorial measure $\mathbf{I}_{\mathsf{R}}$, the k-size minimal inconsistent sets and the l-size minimal inconsistent sets are also not replaceable for any $k \neq l$. Suppose that $k < l$ and M is a k-size minimal inconsistent set, then $\mathbf{I}_{\mathsf{R}}(M) = (0, \cdots, 0, 1) \in \mathbb{R}^k$, then for any l-size minimal inconsistent set M', $\mathbf{I}_{\mathsf{R}}(M') = (0, \cdots, 0, 1) \in \mathbb{R}^l$, so we cannot find a number t s.t. $t\mathbf{I}_{\mathsf{R}}(M') = \mathbf{I}_{\mathsf{R}}(M)$. That is, the contribution made by a k-size minimal inconsistent subset to a belief base cannot be substituted by the contribution made by any finite l-size minimal inconsistent subsets. Then $\mathbf{I}_{\mathsf{R}}$ is a kind of inconsistency measure with *Null Elasticity of Substitution.*

4.2. MinInc Inconsistency Vectorial Value

We have presented properties (D1)-(D5) to characterize an intuitive and discriminative inconsistency measurement for a belief base defined from its minimal inconsistent subsets. Furthermore, to be an intuitive measurement for the blame of each formula of a belief base for the inconsistency of that belief base, an inconsistency (vectorial) value defined from minimal inconsistent subsets, denoted **MIV** , should satisfy the following properties:

(P1) **Innocence**: $\forall \alpha \in K$, $\forall M \in \mathsf{MI}(K)$, $\mathbf{MIV}(M, \alpha) = \mathbf{0}$ if $\alpha \notin M$.

(P2) **Fairness**: $\forall \alpha \in K$, $\forall M \in \mathsf{MI}(K)$, $\mathbf{MIV}(M, \alpha) = \frac{1}{|M|}\mathbf{I}(M)$ if $\alpha \in M$.

(P3) **Cumulation**: $\forall \alpha \in K$, $\mathbf{MIV}(K, \alpha) = \sum\limits_{M \in \mathsf{MI}(K)} \mathbf{MIV}(M, \alpha)$.

The property of Innocence states that any formula not included in a minimal inconsistent subset should not bear any responsibility for the inconsistency of that minimal inconsistent subset. The property of Fairness requires that the blame of each minimal inconsistent subset is shared equally among all the formulas belonging to that minimal inconsistent subset. The property of Cumulation states that the blame of a formula in the inconsistency of a belief base is equal to the sum of the blames of this formula in the inconsistency of all the minimal inconsistent subsets that this formula belongs to.

Note that the property of Innocence may be considered as a more specific statement of the axiom of Minimality presented in (Hunter et al, 2008). Actually, if an inconsistency measure defined from minimal inconsistent subsets accords with the property of Innocence, it must satisfy the axiom of Minimality.

On the other hand, according to the property of Fairness, $\frac{\mathbf{I}_{\mathsf{R}}(M)}{|M|}$ could be considered as a measure of the blame of each formula belonging to the minimal inconsistent belief base M for the inconsistency of M. This motivates the definition of the MinInc Inconsistency Vectorial Measure directly.

Definition 4.4 (MinInc Inconsistency Vectorial Value). Let K be a belief base and B the size of the biggest minimal inconsistent subsets of K. Then the MinInc inconsistency vectorial value $\mathbf{MIV}_{\mathsf{R}}$ is defined as follows:

$$\forall \alpha \in K,\ \mathbf{MIV}_{\mathsf{R}}(K,\alpha) = (r_1, \cdots, r_B),$$

where

$$r_k = \begin{cases} \sum\limits_{M \in \mathsf{MI}^{(k)}(K) s.t. \alpha \in M} \frac{1}{|M|}, & \exists M \in \mathsf{MI}^{(k)}(K) s.t. \alpha \in M; \\ 0, & \text{otherwise.} \end{cases}$$

for each k $(1 \leq k \leq B)$.

Note that $\mathbf{MIV}_{\mathsf{R}}(K,\alpha)$ also assigns $\frac{1}{|M|}$ to each formula belonging to the minimal inconsistent subset M as the blame of each formula for the inconsistency of M. But the relative location of r_k in the vector $(r_1, \cdots, r_B)$ implies the distinction between the blames of the formula for the inconsistency of any two minimal inconsistent subsets with different sizes.

Example 4.4. Consider $K_1 = \{a, \neg a, \neg a \wedge b, c \wedge \neg c, d\}$ again. Then the minimal inconsistent subsets of K_1 are $\mathsf{MI}(K_1) = \langle \mathsf{MI}^{(1)}(K_1), \mathsf{MI}^{(2)}(K_1) \rangle$, where

$$\mathsf{MI}^{(1)}(K_1) = \{\{c \wedge \neg c\}\}, \quad \mathsf{MI}^{(2)}(K_1) = \{\{a, \neg a\}, \{a, \neg a \wedge b\}\}.$$

So, the $\mathbf{MIV}_{\mathsf{R}}$ values are given as follows:

$$\begin{array}{ll} \mathbf{MIV}_{\mathsf{R}}(K_1, a) = (0,1,0) & \mathbf{MIV}_{\mathsf{R}}(K_1, \neg a) = (0, \frac{1}{2}, 0) \\ \mathbf{MIV}_{\mathsf{R}}(K_1, \neg a \wedge b) = (0, \frac{1}{2}, 0) & \mathbf{MIV}_{\mathsf{R}}(K_1, c \wedge \neg c) = (1,0,0) \\ \mathbf{MIV}_{\mathsf{R}}(K_1, d) = (0,0,0) & \end{array}$$

Moreover, $\mathbf{MIV}_{\mathsf{R}}(K_1, a) \neq \mathbf{MIV}_{\mathsf{R}}(K_1, c \wedge \neg c)$. Note that we can not make a distinction between the blame of a and the blame of $c \wedge \neg c$ by using $\mathsf{MIV}_{\mathsf{C}}$.

Proposition 4.3. The MinInc inconsistency vectorial value $\mathbf{MIV}_{\mathsf{R}}$ satisfies (P1), (P2), and (P3), i.e.,

(P1) **Innocence**: $\forall \alpha \in K$, $\forall M \in \mathsf{MI}(K)$, $\mathbf{MIV}_{\mathsf{R}}(M,\alpha) = \mathbf{0}$ if $\alpha \notin M$.

(P2) **Fairness**: $\forall \alpha \in K$, $\forall M \in \mathsf{MI}(K)$, $\mathbf{MIV}_{\mathsf{R}}(M,\alpha) = \frac{1}{|M|}\mathbf{I}_{\mathsf{R}}(M)$ if $\alpha \in M$.

(P3) **Cumulation**: $\forall \alpha \in K$, $\mathbf{MIV}_{\mathsf{R}}(K,\alpha) = \sum\limits_{M \in \mathsf{MI}(K)} \mathbf{MIV}_{\mathsf{R}}(M,\alpha)$.

Clearly, we can get the following results from the definitions of $\mathbf{MIV}_{\mathsf{R}}$ and $\mathsf{MIV}_{\mathsf{C}}$.

Proposition 4.4. Let $\mathsf{MIV_C}$ and $\mathbf{MIV}_\mathsf{R}$ be the MinInc inconsistency value and the MinInc inconsistency vectorial value, respectively. Then

$$\forall \alpha \in K, \mathsf{MIV_C}(K, \alpha) = \mathbf{MIV}_\mathsf{R}(K, \alpha) \cdot (1, 1, \cdots, 1)^\tau,$$

where $(1, 1, \cdots, 1) \in \mathbb{R}^B$, and $(1, 1, \cdots, 1)^\tau$ is the transpose of $(1, 1, \cdots, 1)$.

Proposition 4.5. Let $\mathsf{MIV_C}$ and $\mathbf{MIV}_\mathsf{R}$ be the MinInc inconsistency value and the MinInc inconsistency vectorial value, respectively. Then $\forall \alpha, \beta \in K$,

$$\mathbf{MIV}_\mathsf{R}(K, \alpha) = \mathbf{MIV}_\mathsf{R}(K, \beta) \Longrightarrow \mathsf{MIV_C}(K, \alpha) = \mathsf{MIV_C}(K, \beta).$$

But the converse does not hold.

Example 4.5 (A Counterexample for the Converse). Consider $K_5 = \{a, \neg a, \neg a \wedge c, b \wedge \neg b\}$. Then $\mathsf{MI}(K_5) = \langle \mathsf{MI}^{(1)}(K_5), \mathsf{MI}^{(2)}(K_5) \rangle$, where

$$\mathsf{MI}^{(1)}(K_5) = \{\{b \wedge \neg b\}\}; \quad \mathsf{MI}^{(2)}(K_5) = \{\{a, \neg a\}, \{a, \neg a \wedge c\}\}.$$

We can get

$$\begin{array}{ll} \mathsf{MIV_C}(K_5, a) = 1, & \mathsf{MIV_C}(K_5, b \wedge \neg b) = 1, \\ \mathbf{MIV}_\mathsf{R}(K_5, a) = (0, 1, 0), & \mathbf{MIV}_\mathsf{R}(K_5, b \wedge \neg b) = (1, 0, 0). \end{array}$$

So,

$$\mathsf{MIV_C}(K_5, a) = \mathsf{MIV_C}(K_5, b \wedge \neg b),$$

but

$$\mathbf{MIV}_\mathsf{R}(K_5, a) \prec \mathbf{MIV}_\mathsf{R}(K_5, b \wedge \neg b).$$

The last proposition shows that the MinInc inconsistency vectorial value $\mathbf{MIV}_\mathsf{R}$ is more *discriminative* than the MinInc inconsistency value $\mathsf{MIV_C}$.

Compared to $\sum_{\alpha \in M} \mathsf{MIV_C}(\mathsf{M}, \alpha) = 1$ for a minimal inconsistent set M,

$$\sum_{\alpha \in M} \mathbf{MIV}_\mathsf{R}(M, \alpha) = \mathbf{I}_\mathsf{R}(M).$$

It signifies that $\mathbf{MIV}_\mathsf{R}$ does take the size of M into account. Moreover, it could be used to explain why $\mathbf{MIV}_\mathsf{R}$ is more discriminative than $\mathsf{MIV_C}$.

The MinInc inconsistency vectorial value $\mathbf{MIV}_\mathsf{R}$ also satisfies the following logical properties presented in (Hunter et al, 2008):

Proposition 4.6. Let $\mathbf{MIV}_\mathsf{R}$ be the MinInc inconsistency vectorial value.

–If α is a free formula of K, then $\mathbf{MIV}_\mathsf{R}(K, \alpha) = \mathbf{0}$.
–$\mathbf{MIV}_\mathsf{R}(K, \alpha) \preceq \mathbf{MIV}_\mathsf{R}(K \cup K', \alpha)$.
–If $\alpha \equiv \perp$, then $\mathbf{MIV}_\mathsf{R}(K, \alpha) = (1, 0, \cdots, 0)$.
–If $\phi \vdash \psi$ and $\phi \nvdash \perp$, then $\mathbf{MIV}_\mathsf{R}(K \cup \{\psi\}, \alpha) \preceq \mathbf{MIV}_\mathsf{R}(K \cup \{\phi\}, \alpha)$.

4.3. Shapley Inconsistency Vector Value

We have shown that the MI inconsistency vectorial measure and the MinInc inconsistency vectorial value capture the nature of inconsistency and blame of each formula in the inconsistency for a belief base, respectively. Furthermore, we show that the MinInc inconsistency vectorial value is also a particular kind of

Shapley Inconsistency Value. At first, we adapt the Shapley value to the vectorial form.

Definition 4.5 (Vectorial Coalitional Game). Let $N = \{1, 2, \cdots, n\}$ be a set of n players. Let m be a positive number. A vectorial game in coalitional form is given by a function $\mathbf{v} : 2^N \longrightarrow \mathbb{R}^m$, with $\mathbf{v}(\emptyset) = \mathbf{0}$.

Correspondingly, a value is a function that assigns to each game $\mathbf{v}$ a matrix of payoff $\mathbf{S}(\mathbf{v}) = (\mathbf{S}_1, \mathbf{S}_2, \cdots, \mathbf{S}_n)^\tau$ in $R^{n\times m}$, where $\mathbf{S}_i$ is the payoff for player i.

Definition 4.6 (Shapley Vectorial Value). Let $i \in N$ be a player, and n be the number of players. The Shapley vectorial value of player i in the vectorial game $\mathbf{v}$ is defined as

$$\mathbf{S}_i(\mathbf{v}) = \sum_{C\subseteq N} \frac{(c-1)!(n-c)!}{n!}(\mathbf{v}(C) - \mathbf{v}(C \setminus \{i\}))$$

where c is the cardinality of C.

Evidently, the Shapley vectorial value satisfies all of Efficiency, Symmetry, Dummy and Additivity.

- **Efficiency**: $\sum\limits_{i\in N} \mathbf{S}_i(\mathbf{v}) = \mathbf{v}(N)$.
- **Symmetry**: If i and j are such that for all C s.t. $i, j \notin C$, $\mathbf{v}(C \cup \{i\}) = \mathbf{v}(C \cup \{j\})$, then $\mathbf{S}_i(\mathbf{v}) = \mathbf{S}_j(\mathbf{v})$.
- **Dummy**: If i is such that $\forall C$,$\mathbf{v}(C \cup \{i\}) = \mathbf{v}(C)$, then $\mathbf{S}_i(\mathbf{v}) = \mathbf{0}$.
- **Additivity**: $\mathbf{S}_i(\mathbf{v} + \mathbf{w}) = \mathbf{S}_i(\mathbf{v}) + \mathbf{S}_i(\mathbf{w})$.

On the other hand, for each k $(1 \le k \le m)$, consider the vectorial game $\mathbf{v}_k : 2^N \longrightarrow \mathbb{R}^m$ such that for all $C \subseteq N$, $\mathbf{v}_k(C) = (u_1, \cdots, u_m)$, where $u_i = \begin{cases} v_i & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$ and $\mathbf{v}(C) = (v_1, v_2, \cdots, v_m)$. Let $\mathbf{S}_i(\mathbf{v}_k)$ be the Shapley Vectorial Value for $\mathbf{v}_k$. Then $\mathbf{S}_i(\mathbf{v}_k)$ is the only vectorial value that satisfies all of Efficiency, Symmetry, Dummy and Additivity properties. Suppose that there is another value $\mathbf{S}'_i(\mathbf{v}_k)$ which also satisfies all of Efficiency, Symmetry, Dummy and Additivity properties. Then both $\mathbf{S}_i(\mathbf{v}_k) \cdot (1, 1, \cdots, 1)^\tau$ and $\mathbf{S}'_i(\mathbf{v}_k) \cdot (1, 1, \cdots, 1)^\tau$ are the only Shapley Value of the game v_k such that $v_k(C) = \mathbf{v}_k(C) \cdot (1, 1, \cdots, 1)^\tau$ for all $C \subseteq N$. Then $\mathbf{S}_i(\mathbf{v}_k) = \mathbf{S}'_i(\mathbf{v}_k)$. Furthermore, $\mathbf{S}_i(\mathbf{v}) = \mathbf{S}_i(\mathbf{v}_1) + \cdots + \mathbf{S}_i(\mathbf{v}_m)$. Then we can get the following proposition to support for the Shapley vectorial value.

Proposition 4.7. The Shapley vectorial value is the only value satisfies all of of Efficiency, Symmetry, Dummy and Additivity properties.

Correspondingly, we can define the Shapley Inconsistency Vectorial Value as follows:

Definition 4.7 (Shapley Inconsistency Vectorial Value). Let $\mathbf{I}$ be a basic vectorial inconsistency measure. The corresponding Shapley Inconsistency Vectorial Value (SIVV), denoted $\mathbf{S}^{\mathbf{I}}$, is referred to as the Shapley value of the coalitional game defined by the vectorial function $\mathbf{I}$, i.e. let $\alpha \in K$:

$$\mathbf{S}^{\mathbf{I}}_\alpha(K) = \sum_{C\subseteq N} \frac{(c-1)!(n-c)!}{n!}(\mathbf{I}(C) - \mathbf{I}(C \setminus \{\alpha\}))$$

where n is the cardinality of K and c is the cardinality of C.

Example 4.6. Consider $K_6 = \{a\wedge\neg a, b, \neg b, c\}$. Then $\mathsf{MI}(K_6) = \langle \mathsf{MI}^{(1)}(K_6), \mathsf{MI}^{(2)}(K_6)\rangle$, where

$\mathsf{MI}^{(1)}(K_6) = \{\{a\wedge\neg a\}\}, \quad \mathsf{MI}^{(2)}(K_6) = \{\{b, \neg b\}\}.$

So, we can obtain the MinInc inconsistency vectorial values as follows:

$$\mathbf{MIV}_{\mathsf{R}}(K_6, a\wedge\neg a) = (1, 0), \ \mathbf{MIV}_{\mathsf{R}}(K_6, b) = (0, \frac{1}{2}),$$
$$\mathbf{MIV}_{\mathsf{R}}(K_6, \neg b) = (0, \frac{1}{2}), \ \mathbf{MIV}_{\mathsf{R}}(K_6, c) = (0, 0).$$

Furthermore, we can get the Shapley inconsistency vectorial values as follows:

$$\mathbf{S}^{\mathbf{I}_{\mathsf{R}}}_{a\wedge\neg a}(K_6) = (1, 0), \ \mathbf{S}^{\mathbf{I}_{\mathsf{R}}}_{b}(K_6) = (0, \frac{1}{2}),$$
$$\mathbf{S}^{\mathbf{I}_{\mathsf{R}}}_{\neg b}(K_6) = (0, \frac{1}{2}), \ \mathbf{S}^{\mathbf{I}_{\mathsf{R}}}_{c}(K_6) = (0, 0).$$

Moreover,

$$\forall\alpha \in K_6, \quad \mathbf{S}^{\mathbf{I}_{\mathsf{R}}}_{\alpha}(K_6) = \mathbf{MIV}_{\mathsf{R}}(K_6, \alpha).$$

However, for any belief base, we can prove that the MinInc inconsistency vectorial value $\mathbf{MIV}_{\mathsf{R}}$ is exactly the Shapley Inconsistency Vectorial Value of a coalitional game defined by the MI inconsistency vectorial measure $\mathbf{I}_{\mathsf{R}}$. Firstly, we give the corresponding lemma for the Shapley Inconsistency Vectorial Value.

Lemma 4.1. If a simple vectorial game in coalitional form on a set of players $N = \{1, 2, \cdots, n\}$ is defined by a single winning coalition $C' \subseteq N$, i.e.,

$$\mathbf{v}(C) = \begin{cases} (u_1, \cdots, u_N) & \text{if } C' \subseteq C \\ \mathbf{0} & \text{otherwise} \end{cases}, \quad \text{where } u_i = \begin{cases} 1 & \text{if } \ i = |C'| \\ 0 & \text{if } \ i \neq |C'| \end{cases}.$$

Then the corresponding Shapley vectorial value is:

$$\mathbf{S}_k(\mathbf{v}) = \begin{cases} (s_1, \cdots, s_N) & \text{if } \ k \in C' \\ \mathbf{0} & \text{otherwise} \end{cases}, \quad \text{where } s_i = \begin{cases} \frac{1}{|C'|} & i = |C'| \\ 0 & i \neq |C'| \end{cases}.$$

Based on the lemma above, we can get the following interesting result about $\mathbf{MIV}_{\mathsf{R}}$ and $\mathbf{I}_{\mathsf{R}}$.

Proposition 4.8.

$$\mathbf{S}^{\mathbf{I}_{\mathsf{R}}}_{\alpha}(K) = \mathbf{MIV}_{\mathsf{R}}(K, \alpha).$$

For characterizing the Shapley Inconsistency Vectorial Value, we provide the axiom of Revised MinInc as follows:

– If $M \in \mathsf{MI}(K)$, $\mathbf{I}(M) = \mathbf{I}_{\mathsf{R}}(M)$.

The axiom of revised MinInc accords with the principle of Attenuation under the lexicographical ordering relation as well as the principle of Equal Conflict.

Further, we can get the following axiomatization of the Shapley Inconsistency Vectorial Value.

Proposition 4.9. A Shapley inconsistency vectorial value satisfies Distribution, Symmetry, Minimality, Revised Decomposability and Revised MinInc if and only if it is the MI Shapley Inconsistency vectorial Value $\mathbf{S}^{\mathbf{I}_{\mathsf{R}}}_{\alpha}$.

This proposition shows that the Shapley Inconsistency Vectorial Value $\mathbf{S}_{\alpha}^{\mathbf{I}_{\mathsf{R}}}(K)$ can be axiomatically characterized by the five simple and intuitive properties.

5. Weighted Inconsistency Measures

As revised inconsistency measures, the MI inconsistency vectorial measure $\mathbf{I}_{\mathsf{R}}$ and the MinInc inconsistency vectorial value $\mathbf{MIV}_{\mathsf{R}}$ could be used to measure the inconsistency of a belief base and the blame of each formula in the inconsistency of that belief base, respectively. However, human experts are accustomed to use a single numerical valuation rather than a vector to capture the inconsistency for a belief base in many applications. That is, the human experts are more likely to adopt a weighted value rather than a vector of $(0, \cdots, 0, 1) \in \mathbb{R}^k$ to characterize the amount of inconsistency of a k-size minimal inconsistent set. To address this, in this section, we provide a series of MinInc inconsistency measures from minimal inconsistent subsets in terms of single numerical values.

We have argued that the MI cardinality vector of a belief base can be considered as a more fine-grained characterization of inconsistency in that belief base. Then a numerical inconsistency measure for a belief base K should be a function of $\mathbf{c}(K)$. Particularly, we focus on a series of linear functions of $\mathbf{c}(K)$, i.e., we assign different weights to minimal inconsistent subsets with different sizes.

Suppose that $W = \{w_n\}_{n=1}^{+\infty}$ is a sequence of real numbers such that

(A1) $\forall n \in \mathbb{N}$, $w_n > 0$;
(A2) $\forall n, m \in \mathbb{N}$, $w_n > w_m$ if $n < m$;
(A3) $\lim\limits_{n \to +\infty} w_n = 0$.

Further, we use $\mathbf{w}_n$ to denote a n-size vector $(w_1, \cdots, w_n)$ for each $n \in \mathbb{N}$. Then we define the weighted MI inconsistency measure as follows:

Definition 5.1 (Weighted MI inconsistency Measure). Let K be a belief base and B the size of the largest minimal inconsistent subsets of K. Let $\mathbf{I}_{\mathsf{R}}$ be the MI inconsistency vectorial measure. Then the weighted MI inconsistency measure for K, denoted $\mathsf{I}_{\mathsf{W}}(K)$, is defined as follows:

$$\mathsf{I}_{\mathsf{W}}(K) = \mathbf{I}_{\mathsf{R}}(K) \cdot \mathbf{w}_B^{\tau},$$

where $\mathbf{w}_B^{\tau}$ is the transpose of $\mathbf{w}_B$.

Note that $\forall M \in \mathsf{MI}(K)$, $\mathsf{I}_{\mathsf{W}}(M) = w_{|M|}$. This is the reason why we termed I_{W} the weighted MI inconsistency measure.

The weighted MI inconsistency measure satisfies all the constraints mentioned above.

Proposition 5.1. Let I_{W} be a weighted MI inconsistency measure. Then I_{W} satisfies the following desiderata:

(D1) **Consistency**: $\mathsf{I}_{\mathsf{W}}(K) = 0$ iff $\mathsf{MI}(K) = \emptyset$.
(D2) **Monotony w.r.t. MI**: $\mathsf{I}_{\mathsf{W}}(K_1) \leq \mathsf{I}_{\mathsf{W}}(K_2)$ if $\mathsf{MI}(K_1) \subseteq \mathsf{MI}(K_2)$.
(D3) **Attenuation**: For any two minimal inconsistent sets M_1 and M_2, $\mathsf{I}_{\mathsf{W}}(M_1) < \mathsf{I}_{\mathsf{W}}(M_2)$ if $|M_1| > |M_2|$.
(D4) **Equal Conflict**: For any two minimal inconsistent sets M_1 and M_2, $\mathsf{I}_{\mathsf{W}}(M_1) = \mathsf{I}_{\mathsf{W}}(M_2)$ if $|M_1| = |M_2|$.

(D5) **Separability**: $\mathsf{I}_\mathsf{W}(K_1 \cup K_2) = \mathsf{I}_\mathsf{W}(K_1) + \mathsf{I}_\mathsf{W}(K_2)$ if $\mathsf{MI}(K_1 \cup K_2) = \mathsf{MI}(K_1) \oplus \mathsf{MI}(K_2)$.

Moreover, the weighted MI inconsistency measure satisfies some other intuitive properties.

Proposition 5.2. The weighted MI inconsistency measure I_W is linearly homogeneous:

$$\mathsf{I}_\mathsf{W}(K_2) = t\mathsf{I}_\mathsf{W}(K_1) \text{ if } \mathbf{c}(K_2) = t\mathbf{c}(K_1).$$

Proposition 5.3. Let I_W be a weighted MI inconsistency measure and M a minimal inconsistent belief base. Then

–**Inconsistency**: $\mathsf{I}_\mathsf{W}(M) > 0$;
–**Maximal Contradiction**: $\mathsf{I}_\mathsf{W}(M) = w_1$ if $|M| = 1$.
–**Almost consistency**: $\lim\limits_{|M| \to +\infty} \mathsf{I}_\mathsf{W}(M) = 0$.

The Inconsistency property states that any minimal inconsistent set has non-zero amount of inconsistency. The Maximal Contradiction states that the inconsistency of any minimal inconsistent belief base is no more than that of a singleton set of a contradictory formula (i.e. w_1). It accords with the property of Attenuation. The Almost Consistency property states that as the size of minimal inconsistent set increases, the amount of inconsistency tends to zero.

Correspondingly, we define the weighted MinInc inconsistency value as follows:

Definition 5.2 (Weighted MinInc inconsistency Value). Let K be a belief base and B the size of the largest minimal inconsistent subsets of K. Let $\mathbf{MIV}_\mathsf{R}$ be the MinInc inconsistency vectorial value. Then the weighted MinInc inconsistency value for K, denoted MIV_W, is defined as follows:

$$\forall \alpha \in K, \quad \mathsf{MIV}_\mathsf{W}(K, \alpha) = \mathbf{MIV}_\mathsf{R}(K, \alpha) \cdot \mathbf{w}_B^\tau.$$

where $\mathbf{w}_B^\tau$ is the transpose of $\mathbf{w}_B$.

Clearly, the weighted MinInc inconsistency value satisfies the basic properties about measurements for the blame of each formula for the inconsistency of a belief base.

Proposition 5.4. The weighted MinInc inconsistency value MIV_W satisfies (P1), (P2), and (P3), i.e.,

(P1) **Innocence**: $\forall \alpha \in K$, $\forall M \in \mathsf{MI}(K)$, $\mathsf{MIV}_\mathsf{W}(M, \alpha) = 0$ if $\alpha \notin M$.
(P2) **Fairness**: $\forall \alpha \in K$, $\forall M \in \mathsf{MI}(K)$, $\mathsf{MIV}_\mathsf{W}(M, \alpha) = \frac{1}{|M|}\mathsf{I}_\mathsf{W}(M)$ if $\alpha \in M$.
(P3) **Cumulation**: $\mathsf{MIV}_\mathsf{W}(K, \alpha) = \sum\limits_{M \in \mathsf{MI}(K)} \mathsf{MIV}_\mathsf{W}(M, \alpha)$.

If a sequence W satisfies the property (A2), i.e., $\forall n, m \in \mathbb{N}$, $w_n > w_m$ if $n < m$, then the weighted measures based on W support the property of Attenuation about a revised MI inconsistency measure. Obviously, there is no sequence W s.t. $w_n - w_{n+1} = c$, where c is a constant and $0 < c < w_1$. That is, the amount of inconsistency in a minimal inconsistent set cannot decrease by deleting a constant increment as the size increases. Then we consider the ratio of any two successive weights. For each weight sequence W, we define a proportion function $P_W : \mathbb{N} \to \mathbb{R}$ such that $P_W(n) = \frac{w_{n+1}}{w_n}$ for each $n \in \mathbb{N}$.

Definition 5.3 (Type-I Weight Sequence). A weight sequence W is called a Type-I weight sequence if

$$\forall n \in \mathbb{N}, \quad P_W(n) = \lambda, \text{ where } \lambda \text{ is a constant and } 0 < \lambda < 1.$$

This implies that the amount of a minimal inconsistent set attenuates at a constant proportion λ as the size of the minimal inconsistent set increases.

A Type-I weight sequence can be represented by $W^{\mathrm{I}} = \{a\lambda^{n-1}\}_{n=1}^{+\infty}$, where a (i.e. w_1^{I}) is a positive constant. Then a Type-I weighted MI inconsistency measure and weighted MinInc inconsistency value can be defined as follows:

Definition 5.4 (Type-I Weighted MI Inconsistency Measure). Let K be a belief base and B the size of the largest minimal inconsistent subsets of K. Let $\mathbf{I}_{\mathsf{R}}$ be the MI inconsistency vectorial measure. Then the Type-I weighted MI inconsistency measure for K, denoted $\mathsf{I}_{\mathsf{W}^{\mathrm{I}}}(K)$, is defined as follows:

$$\mathsf{I}_{\mathsf{W}^{\mathrm{I}}}(K) = \mathbf{I}_{\mathsf{R}}(K) \cdot (\mathbf{w}_B^{\mathrm{I}})^{\tau} = \sum_{n=1}^{B} a\lambda^{n-1}|\mathsf{MI}^{(n)}(K)|$$

where constants a and λ satisfy $a > 0$ and $0 < \lambda < 1$, respectively.

Definition 5.5 (Type-I Weighted MinInc Inconsistency Value). Let K be a belief base and B the size of the largest minimal inconsistent subsets of K. Let $\mathbf{MIV}_{\mathsf{R}}$ be the MinInc inconsistency vectorial value. Then the Type-I weighted MinInc inconsistency value for K, denoted $\mathsf{MIV}_{\mathsf{W}^{\mathrm{I}}}$, is defined as follows: $\forall \alpha \in K$,

$$\mathsf{MIV}_{\mathsf{W}^{\mathrm{I}}}(K, \alpha) = \mathbf{MIV}_{\mathsf{R}}(K, \alpha) \cdot (\mathbf{w}_B^{\mathrm{I}})^{\tau} = \sum_{M \in \mathsf{MI}(K) s.t. \alpha \in M} \frac{a\lambda^{|M|-1}}{|M|}.$$

where constants a and λ satisfy $a > 0$ and $0 < \lambda < 1$, respectively.

Example 5.1. Consider $K_7 = \{a \wedge \neg a, b, \neg b, c \wedge \neg b, c\}$ again. Then $\mathsf{MI}(K_7) = \langle \mathsf{MI}^{(1)}(K_7), \mathsf{MI}^{(2)}(K_7) \rangle$, where

$$\mathsf{MI}^{(1)}(K_7) = \{\{a \wedge \neg a\}\},$$
$$\mathsf{MI}^{(2)}(K_7) = \{\{b, \neg b\}, \{b, c \wedge \neg b\}\}.$$

Suppose that we use a Type-I weight sequence W^{I}, in which $a = 1$ and $\lambda = \frac{1}{e}$. So,

$$\mathsf{I}_{\mathsf{W}^{\mathrm{I}}}(K_7) = \tfrac{e+2}{e}, \quad \mathsf{I}_{\mathsf{W}^{\mathrm{I}}}(\{a \wedge \neg a\}) = 1,$$
$$\mathsf{I}_{\mathsf{W}^{\mathrm{I}}}(\{b, \neg b\}) = \tfrac{1}{e}, \quad \mathsf{I}_{\mathsf{W}^{\mathrm{I}}}(\{b, c \wedge \neg b\}) = \tfrac{1}{e}.$$

We can obtain the Type-I weighted MinInc inconsistency values as follows:

$$\mathsf{MIV}_{\mathsf{W}^{\mathrm{I}}}(K_7, a \wedge \neg a) = 1, \quad \mathsf{MIV}_{\mathsf{W}^{\mathrm{I}}}(K_7, b) = \tfrac{1}{e},$$
$$\mathsf{MIV}_{\mathsf{W}^{\mathrm{I}}}(K_7, \neg b) = \tfrac{1}{2e}, \quad \mathsf{MIV}_{\mathsf{W}^{\mathrm{I}}}(K_7, c \wedge \neg b) = \tfrac{1}{2e},$$
$$\mathsf{MIV}_{\mathsf{W}^{\mathrm{I}}}(K_7, c) = 0.$$

Note that $\mathsf{MIV}_{\mathsf{C}}(K_7, b) = \mathsf{MIV}_{\mathsf{C}}(K_7, a \wedge \neg a)$. But $\mathsf{MIV}_{\mathsf{W}^{\lambda}}(K_7, b) < \mathsf{MIV}_{\mathsf{W}^{\lambda}}(K_7, a \wedge \neg a)$. It signifies that $\mathsf{MIV}_{\mathsf{W}^{\lambda}}$ can differentiate the blame of $a \wedge \neg a$ and the blame of b in inconsistency of K_7.

Definition 5.6 (Type-II Weight Sequence). A weight sequence W is called a Type-II weight sequence if

(1) $\forall n \in \mathbb{N},\ P_W(n) < P_W(n+1)$;
(2) $\lim_{n \to +\infty} P_W(n) = 1$.

Note that Condition (1) states that the ratio of two successive weights $P_W(n)$ increases as n increases. Moreover, Condition (2) states that the ratio of two successive weights tends to 1.

Definition 5.7. The weight sequence $W^{(\mathrm{c})}$ is defined as follows:

$$\forall n \in \mathbb{N},\quad w_n^{(\mathrm{c})} = \frac{1}{n}.$$

Obviously, $W^{(\mathrm{c})}$ is a Type-II weight sequence. Correspondingly, we define a Type-II weighted MI inconsistency measure and weighted MinInc inconsistency value as follows:

Definition 5.8. Let K be a belief base and B the size of the largest minimal inconsistent subsets of K. A Type-II weighted MI inconsistency measure based on $W^{(\mathrm{c})}$ for K, denoted $\mathsf{I}_{\mathsf{W}^{(\mathrm{c})}}(K)$, is defined as follows:

$$\mathsf{I}_{\mathsf{W}^{(\mathrm{c})}}(K) = \mathbf{I}_{\mathsf{R}}(K) \cdot (\mathbf{w}_B^{(\mathrm{c})})^{\tau} = \sum_{n=1}^{B} \frac{|\mathsf{MI}^{(n)}(K)|}{n}.$$

Definition 5.9. Let K be a belief base and B the size of the largest minimal inconsistent subsets of K. A Type-II weighted MinInc inconsistency value based on $W^{(\mathrm{c})}$ for K, denoted $\mathsf{MIV}_{\mathsf{W}^{(\mathrm{c})}}$, is defined as follows: $\forall \alpha \in K$,

$$\mathsf{MIV}_{\mathsf{W}^{(\mathrm{c})}}(K, \alpha) = \mathbf{MIV}_{\mathsf{R}}(K, \alpha) \cdot (\mathbf{w}_B^{(\mathrm{c})})^{\tau} = \sum_{M \in \mathsf{MI}(K) s.t. \alpha \in M} \frac{1}{|M|^2}.$$

Example 5.2. Consider $K_2 = \{a, a \to b, \neg b, \neg a, c \wedge \neg c\}$ again. Then $B = 3$. Suppose we use a Type-II weight sequence $W^{(\mathrm{c})} = \{\frac{1}{n}\}_{n=1}^{+\infty}$. We then get $\mathsf{I}_{\mathsf{W}^{(\mathrm{c})}}$ for K and minimal inconsistent subsets of K as follows:

$\mathsf{I}_{\mathsf{W}^{(\mathrm{c})}}(K_2) = \frac{11}{6}$, $\quad \mathsf{I}_{\mathsf{W}^{(\mathrm{c})}}(\{c \wedge \neg c\}) = 1$,
$\mathsf{I}_{\mathsf{W}^{(\mathrm{c})}}(\{a, \neg a\}) = \frac{1}{2}$, $\quad \mathsf{I}_{\mathsf{W}^{(\mathrm{c})}}(\{a, a \to b, \neg b\}) = \frac{1}{3}$.

$$\mathsf{I}_{\mathsf{W}^{(\mathrm{c})}}(\{a, a \to b, \neg b\}) < \mathsf{I}_{\mathsf{W}^{(\mathrm{c})}}(\{a, \neg a\}) < \mathsf{I}_{\mathsf{W}^{(\mathrm{c})}}(\{c \wedge \neg c\}).$$

Furthermore, we get the $\mathsf{MIV}_{\mathsf{W}^{(\mathrm{c})}}$ values for K as follows:

$$\mathsf{MIV}_{\mathsf{W}^{(\mathrm{c})}}(K_2, c \wedge \neg c) = (1,0,0)(1, \frac{1}{2}, \frac{1}{3})^{\tau} = 1,$$
$$\mathsf{MIV}_{\mathsf{W}^{(\mathrm{c})}}(K_2, a) = (0, \frac{1}{2}, \frac{1}{3})(1, \frac{1}{2}, \frac{1}{3})^{\tau} = \frac{13}{36},$$
$$\mathsf{MIV}_{\mathsf{W}^{(\mathrm{c})}}(K_2, \neg a) = (0, \frac{1}{2}, 0)(1, \frac{1}{2}, \frac{1}{3})^{\tau} = \frac{1}{4},$$
$$\mathsf{MIV}_{\mathsf{W}^{(\mathrm{c})}}(K_2, a \to b) = (0, 0, \frac{1}{3})(1, \frac{1}{2}, \frac{1}{3})^{\tau} = \frac{1}{9},$$
$$\mathsf{MIV}_{\mathsf{W}^{(\mathrm{c})}}(K_2, \neg b) = (0, 0, \frac{1}{3})(1, \frac{1}{2}, \frac{1}{3})^{\tau} = \frac{1}{9}.$$

However, the MI inconsistency measure I_{MI} can be considered as a particular weighted MI inconsistency measure, in which the weight sequence is $\{w_n =$

$1\}_{n=1}^{+\infty}$. Obviously, this weight sequence does not satisfy the following properties of a required weight sequence,

(A2) $w_n > w_m$ if $n < m$,
(A3) $\lim_{n\to+\infty} w_n = 0$.

Then $\mathsf{I_{MI}}$ cannot satisfy the Attenuation and Almost Consistency properties. This distinguishes $\mathsf{I_{MI}}$ from $\mathsf{I_W}$. Actually, the following proposition shows that $\mathsf{I_W}$ is more discriminative than $\mathsf{I_{MI}}$.

Proposition 5.5. Let M_1 and M_2 be two minimal inconsistent belief bases. Then

$$\mathsf{I_W}(M_1) = \mathsf{I_W}(M_2) \Longrightarrow \mathsf{I_{MI}}(M_1) = \mathsf{I_{MI}}(M_2).$$

But the converse does not hold.

Given a belief base K, $\forall M \in \mathsf{MI}(K)$, $\mathsf{MIV_W}(M,\alpha) = w_{|M|}\mathsf{MIV_C}(M,\alpha)$. Then $\sum_{\alpha\in M} \mathsf{MIV_W}(M,\alpha) = w_{|M|}$. It means that the weighted MinInc inconsistency value takes the attenuation of inconsistency into account.

One of the most significant results of our approach is that the weighted MinInc inconsistency value $\mathsf{MIV_W}$ is exactly the Shapley Inconsistency Value of a coalitional game defined by the weighted MI inconsistency measure $\mathsf{I_W}$.

Proposition 5.6.

$$S_\alpha^{\mathsf{I_W}}(K) = \mathsf{MIV_W}(K,\alpha).$$
$$S_\alpha^{\mathsf{I_{W^I}}}(K) = \mathsf{MIV_{W^I}}(K,\alpha).$$
$$S_\alpha^{\mathsf{I_{W^{(c)}}}}(K) = \mathsf{MIV_{W^{(c)}}}(K,\alpha).$$

Furthermore, if we provide an axiom of Weighted MinInc as follows:

– If $M \in \mathsf{MI}(K)$, then $\mathsf{I}(M) = w_{|M|}$.

Then the Shapley Inconsistency Value $\mathbf{S}_\alpha^{\mathsf{I_W}}(K)$ can be axiomatically characterized by the five simple and intuitive properties, as shown in **Proposition** 5.7.

Proposition 5.7. A Shapley inconsistency value satisfies Distribution, Symmetry, Minimality, Revised Decomposability and Weighted MinInc if and only if it is $\mathbf{S}_\alpha^{\mathsf{I_W}}$.

As mentioned above, both the MI inconsistency vectorial measure and the MinInc inconsistency vectorial value are measures with null elasticity of substitution. However, the null elasticity of substitution makes the MI inconsistency vectorial measure and the MinInc inconsistency vectorial value more discriminating.

The weighted MI inconsistency measure and the weighted MinInc inconsistency value are partially replaceable. For example, if we use the weight sequence $W^{(c)} = \{\frac{1}{n}\}_{n=1}^{+\infty}$, then $w_1 = n \cdot w_n$ for each $n \in \mathbb{N}$. It signifies that the amount of inconsistency in a minimal inconsistent singleton set is equal to the total amount of inconsistency in n minimal inconsistent subsets each with size n. That is, the weighted measures weaken the distinction of minimal inconsistent subsets with different sizes. Then the weighted measures may be less discriminating than the vectorial measures. Actually, we have following results:

Proposition 5.8. Let I_{W} and $\mathbf{I}_{\mathsf{R}}$ be the weighted MI inconsistency measure and the MI inconsistency vectorial measure, respectively. Then $\forall K_1, K_2 \in \mathcal{K}_{\mathcal{L}}$,

$$\mathbf{I}_{\mathsf{R}}(K_1) = \mathbf{I}_{\mathsf{R}}(K_2) \implies \mathsf{I}_{\mathsf{W}}(K_1) = \mathsf{I}_{\mathsf{W}}(K_2).$$

But the converse does not hold for some W.

Example 5.3 (A Counterexample for the Converse). Consider $K_8 = \{a \wedge \neg a\}$ and $K_9 = \{a, \neg b, b, \neg a\}$. Then

$$\mathsf{MI}(K_8) = \mathsf{MI}^{(1)}(K_8) = \{\{a \wedge \neg a\}\},$$
$$\mathsf{MI}(K_9) = \mathsf{MI}^{(2)}(K_9) = \{\{b, \neg b\}, \{a, \neg a\}\}.$$

So,

$$\mathsf{I}_{\mathsf{W}^{(c)}}(K_8) = 1, \quad \mathsf{I}_{\mathsf{W}^{(c)}}(K_9) = 1;$$
$$\mathbf{I}_{\mathsf{R}}(K_8) = (1), \quad \mathbf{I}_{\mathsf{R}}(K_9) = (0, 2).$$

Evidently,

$$\mathsf{I}_{\mathsf{W}^{(c)}}(K_8) = \mathsf{I}_{\mathsf{W}^{(c)}}(K_9).$$

But

$$\mathbf{I}_{\mathsf{R}}(K_9) \prec \mathbf{I}_{\mathsf{R}}(K_8).$$

Proposition 5.9. Let $\mathsf{MIV}_{\mathsf{W}}$ and $\mathbf{MIV}_{\mathsf{R}}$ be the weighted MinInc inconsistency value and the MinInc inconsistency vectorial value, respectively. Then $\forall \alpha, \beta \in K$,

$$\mathbf{MIV}_{\mathsf{R}}(K, \alpha) = \mathbf{MIV}_{\mathsf{R}}(K, \beta) \Longrightarrow \mathsf{MIV}_{\mathsf{W}}(K, \alpha) = \mathsf{MIV}_{\mathsf{W}}(K, \beta).$$

But the converse does not hold for some W.

Example 5.4 (A Counterexample for the Converse). Consider $K_{10} = \{a \wedge \neg a, b, \neg b, \neg b \wedge c, \neg b \wedge d, \neg b \wedge a\}$. Then $\mathsf{MI}(K_{10}) = \langle \mathsf{MI}^{(1)}(K_{10}), \mathsf{MI}^{(2)}(K_{10}) \rangle$, where

$$\mathsf{MI}^{(1)}(K_{10}) = \{\{a \wedge \neg a\}\};$$
$$\mathsf{MI}^{(2)}(K_{10}) = \{\{b, \neg b\}, \{b, \neg b \wedge c\}, \{b, \neg b \wedge d\}, \{b, \neg b \wedge a\}\}.$$

We can get

$$\mathsf{MIV}_{\mathsf{W}^{(c)}}(K_{10}, a \wedge \neg a) = 1, \qquad \mathsf{MIV}_{\mathsf{W}^{(c)}}(K_{10}, b) = 1,$$
$$\mathbf{MIV}_{\mathsf{R}}(K_{10}, a \wedge \neg a) = (1, 0), \qquad \mathbf{MIV}_{\mathsf{R}}(K_{10}, b) = (0, 2).$$

So,

$$\mathsf{MIV}_{\mathsf{W}^{(c)}}(K_{10}, a \wedge \neg a) = \mathsf{MIV}_{\mathsf{W}^{(c)}}(K_{10}, b),$$

but

$$\mathbf{MIV}_{\mathsf{R}}(K_{10}, b) \prec \mathbf{MIV}_{\mathsf{R}}(K_{10}, a \wedge \neg a).$$

However, how to select an appropriate weight sequence to construct more discriminating weighted inconsistency measures is really an important but open issue. Generally, it depends on application domains. Moreover, a good weight sequence should emphasize the distinction between minimal inconsistent subsets with different sizes, i.e, it should help the corresponding weighted measures retaining discriminative characteristics of the MI inconsistency vectorial measure as much as possible. For example, $(0, 0, 1, 0)$ will never be equal to $(0, 0, 0, n)$ for any n, so we want $(0, 0, 1, 0)(w_1, \cdots, w_4)^{\tau} \neq (0, 0, 0, n)(w_1, \cdots, w_4)^{\tau}$ being true if possible. In this sense, for example, $W^{\frac{1}{e}} = \{e^{1-n}\}_{n=1}^{+\infty}$ maybe better than $W^{\frac{1}{2}} = \{2^{1-n}\}_{n=1}^{+\infty}$ for defining weighted MI inconsistency measure I_{W}, since the amount of inconsistency of a single n-size minimal inconsistent subset can be replaced by that of two $n+1$-size minimal inconsistent subsets if we use $W^{\frac{1}{2}}$. In contrast, if we use $W^{\frac{1}{e}}$, then the amount of inconsistency of a single n-size minimal inconsistent subset cannot be replaced by that of a number of $n+1$-size minimal inconsistent subsets.

6. Related Work

We have presented two kinds of revised inconsistency measure for a belief base by using the minimal inconsistent subsets of that belief base. However, measuring inconsistency has received considerable attention in computer science as well as artificial intelligence recently. In this section, we compare the revised inconsistency measure presented in this paper with some of closely related research.

A number of proposals for measuring the degree of inconsistency of a belief base have been presented recently. These proposals for measuring inconsistency have been classified into two approaches in (Hunter et al, 2006; Hunter et al, 2008). The proposals of the first approach always focus on counting the minimal number of formulas needed to cause an inconsistency in a set of formulas, such as the measurement of maximal $\frac{|M|-1}{|M|}$-consistency for a minimal inconsistent set M presented in (Knight, 2002). This approach supports an intuition that the more formulas needed (in a set) to cause an inconsistency, the less inconsistent is the set (Knight, 2002). Evidently, the MI inconsistency vectorial measure and the weighted MI inconsistency measure presented in this paper can be considered as particular proposals of the first approach, and both the two revised measures support this intuition. Note that the two kinds of revised inconsistency measures presented in this paper are syntax sensitive. As argued in (Hunter et al, 2008), the syntax sensitivity is necessary in some applications such as requirements engineering.

The proposals of the second approach supports the intention of looking inside the formulas (Hunter et al, 2006; Hunter et al, 2008), such as model-based proposals (Hunter, 2002; Konieczny et al, 2003; Grant et al, 2006; Grant and Hunter, 2008). As pointed out in (Hunter et al, 2006; Hunter et al, 2008), the first approach rejects the possibility of a finer-grained inspection of the formulas, whilst the second approach does not consider the distribution of the contradiction among the formulas because of lack of syntax sensitivity.

The Shapley inconsistency value presented in (Hunter et al, 2006) can be considered as the first attempt to build measures that allow us to take the best of the two approaches. Roughly speaking, an inconsistency measure of the second approach is used to define a game in coalitional form firstly, then the contradictions can be distributed to each formula by using Shapley value model. More importantly, the Shapley inconsistency value can be considered as an useful pattern to link the inconsistency measures for subsets of a belief base and the inconsistency measures for each formula in inconsistency of that belief base (Hunter et al, 2008). In this paper, we also use the model of Shapley value to characterize the MinInc inconsistency vectorial value as well as the weighted MinInc inconsistency value. That is, the two kinds of revised MinInc inconsistency values presented in this paper have been shown to be particular Shapley inconsistency values.

On the other hand, it is increasingly recognized that the minimal inconsistent subsets of a belief base can be used to derive an intuitive evaluation of the amount of inconsistency for the base. The scoring function presented in (Hunter, 2004) can be considered as an earlier representative measurement in terms of minimal inconsistent subsets. Given a belief base K, for each subset K' of K, the scoring function $S(K')$, defined as $|\mathsf{MI}(K)| - |\mathsf{MI}(K')|$, gives the number of minimal inconsistent subsets of K that would be eliminated if K' was removed from K. Especially, for each singleton set $\{\alpha\}$, $S(\{\alpha\})$ counts the number of minimal inconsistent subsets that α belongs to. Then it may be considered as a measurement for the blame of α in the inconsistency of K in some sense. However, as pointed out in (Hunter et al, 2008), the idea is very sketchy, since it does not consider the size of each minimal inconsistent subset α belongs to. For example, α has the same contribution to $\{\alpha, \neg\alpha\}$ and $\{\alpha, \beta \rightarrow \neg\alpha, \beta\}$ according to the scoring function.

To address this, the MinInc inconsistency value $\mathsf{MIV_C}$ presented in (Hunter et al, 2008) have taken the size of each minimal inconsistent subset into account in order to evaluate the inconsistency value of each formula. In detail, for each minimal inconsistent subset M that α belongs to, $\frac{1}{|M|}$ rather than 1 is considered as the evaluation of the blame of α in the inconsistency in M. On the other hand, $\sum\limits_{\alpha \in M} \mathsf{MIV_C}(M, \alpha) = 1$ implies that each minimal inconsistent subset has the same amount of inconsistency, which is termed as the axiom of MinInc (Hunter et al, 2008). However, as illustrated by the lottery paradox, intuitively, as the size of a minimal inconsistent subset increases, the amount of inconsistency attenuates (Knight, 2002). The MinInc inconsistency value fails to support this intuition. Furthermore, to characterize the MinInc inconsistency value by using Shapley Value model, a basic inconsistency measure for a belief base, termed the MI inconsistency measure $\mathsf{I_{MI}}$, has also been defined from minimal inconsistent subsets. However, most properties used to characterize a basic inconsistency measure are too general for measures defined from minimal inconsistent subsets. That is, most properties do not take the characteristics of minimal inconsistent subsets into account.

In contrast, we presented five properties (i.e., Consistency, Monotony w.r.t. MI, Attenuation, Equal conflict, Separability) to characterize an intuitive inconsistency measure for belief bases defined from minimal inconsistent subsets. In particular, *Attenuation* and *Equal conflict* describe the relation of the amount of inconsistency in a minimal inconsistent subset and the size of that minimal inconsistent subset. Moreover, we presented three properties (Innocence, Fairness, Cumulation) to characterize an intuitive inconsistency measure for the blame of each formula in the inconsistency of a belief base.

Then we provided two kinds of revised MI inconsistency measure and the corresponding revised MinInc inconsistency values for a belief base. Both of the two kinds of the revised measures accord with the intuition illustrated by the lottery paradox: the smaller the size of a minimal inconsistent subset, the bigger the inconsistency is. In detail, we firstly present the MinInc inconsistency value $\mathbf{MIV}_{\mathsf{R}}$, in which a vector $(0, \cdots, 0, \frac{1}{|M|}) \in \mathbb{R}^{|M|}$ rather than $\frac{1}{|M|}$ is used to evaluate the blame of each formula of M in the inconsistency. In this vector, as the MinInc inconsistency value, the numerical value $\frac{1}{|M|}$ implies that each formula of M only accounts for $\frac{1}{|M|}$ of cause of the inconsistency in M. But the location of $\frac{1}{|M|}$ in the vector implies that the total amount of inconsistency in M depends on the size of M. That is, $\sum_{\alpha \in M} \mathbf{MIV}_{\mathsf{R}}(M, \alpha) = \mathbf{I}_{\mathsf{R}}(M) = (0, \cdots, 0, 1) \in \mathbb{R}^{|M|}$. It distinguishes the MinInc inconsistency vectorial value from the MinInc inconsistency value. Furthermore, we demonstrated that the MinInc inconsistency vectorial value is more discriminative than the MinInc inconsistency value presented in (Hunter et al, 2008).

We then presented the weighted MinInc inconsistency value $\mathsf{MIV_W}$ to support the demand for measuring the inconsistency of a belief base by using a single numerical value. Given a minimal inconsistent belief base M, $\frac{1}{|M|} \cdot w_{|M|}$ is used to evaluate the blame of each formula in the inconsistency of M according to $\mathsf{MIV_W}$. Note that the factor $\frac{1}{|M|}$ also signifies that each formula in M only accounts for $\frac{1}{|M|}$ of the cause of inconsistency in M. But the weight $w_{|M|}$ states that the total amount of inconsistency in M is captured by $w_{|M|}$ rather than a vector. Moreover, according to the property (A2) of the weight sequence, $w_{|M|}$ attenuates as $|M|$ increases. This means the weighted inconsistency measures $\mathsf{I_W}$ and $\mathsf{MIV_W}$ also overcome the disadvantages of $\mathsf{I_{MI}}$ and $\mathsf{MIV_C}$.

7. Conclusions

It is increasingly recognized that it is natural to explore relationships between measures of inconsistency for a belief base and the minimal inconsistent subsets of that belief base. A. Hunter and S. Konieczny have proposed an inconsistency value termed $\mathsf{MIV_C}$ from minimal inconsistent subsets, and have shown that it can be axiomatized completely in terms of five simple axioms (Hunter et al, 2008). However, as we pointed out in this paper, the axiom of MinInc used to characterize the value $\mathsf{MIV_C}$ does not support the intuition that as the size of minimal inconsistent set increases, the degree of inconsistency becomes more tolerable.

We presented two kinds of revised MinInc inconsistency value from the minimal inconsistent subsets, which consider the size of each minimal inconsistent subset as well as the number of minimal inconsistent subsets. We first explored the intuitive and simple properties to characterize inconsistency measures defined from minimal inconsistent subsets. We then provided the first kind of revised measures in terms of vector, i.e., the MI inconsistency vectorial measure and the MinInc inconsistency vectorial value. We demonstrated that the two revised measures are more discriminating than the corresponding measures defined in (Hunter et al, 2008). Finally, based on the vectorial revised measures, we provided the weighted MI inconsistency measure and the weighted MinInc inconsistency value.

We have shown that both the two kinds of revised measures support the intuition that the smaller the size of the minimal inconsistent subset, the bigger the inconsistency is. More importantly, we have also proved that both of the two kinds of revised MinInc inconsistency values are particular Shapley inconsistency values with regard to corresponding revised MI inconsistency values, respectively. Moreover, the two kinds of revised MinInc inconsistency values can be axiomatically characterized by the revised Decomposability and MinInc axioms and three other simple properties.

We focused on the flat belief base in this paper. However, the reality is that some beliefs are more important than others. How to measure the inconsistency for a stratified belief base from minimal inconsistent subsets will be the main direction for our future work.

Acknowledgements. The authors are grateful to reviewers for their valuable comments. This work was partly supported by the National Natural Science Foundation of China under Grant No. 60703061, the National Key Research and Development Program of China under Grant No. 2009CB320701, the Key Project of National Natural Science Foundation of China under Grant No. 90818026, and the National Natural Science Fund for Distinguished Young Scholars of China under Grant No.60625204, and the NSFC and the British Royal Society China-UK Joint Project.

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Appendix

The Proof of Proposition 4.8

Proof: First suppose that α is a free formula of K, then we get $\mathbf{S}_\alpha^{\mathbf{I}_{\mathsf{R}}}(K) = \mathbf{0}$ by Dummy and Minimality. By the definition we know that $\mathbf{MIV}_{\mathsf{R}}(K, \alpha) = \mathbf{0}$. So the proposition holds in this case.

Then suppose that α is not a free formula of K. Notice that $\mathbf{I}_{\mathsf{R}}(C)$ can be decomposed in $\mathbf{I}_{\mathsf{R}}(C) = \sum_{M \in \mathsf{MI}(K)} \mathbf{v}_M(C)$ such that

$$\mathbf{v}_M(C) = \begin{cases} (u_1, \cdots, u_m) & \text{if } M \subseteq C \\ \mathbf{0} & \text{otherwise} \end{cases}, \text{ where } u_i = \begin{cases} 1 & \text{if } i = |M| \\ 0 & \text{if } i \neq |M| \end{cases}.$$

We use $\mathbf{v}_M(K)$ to denote the vectorial game in coalitional form defined from K and $\mathbf{v}_M$. Then by the Lemma we have

$$\mathbf{S}_\alpha(\mathbf{v}_M) = \begin{cases} (s_1, \cdots, s_m) & \text{if } \alpha \in M \\ \mathbf{0} & \text{if } \alpha \notin M \end{cases}, \text{ where } s_i = \begin{cases} \frac{1}{|M|} & \text{if } i = |M| \\ 0 & \text{if } i \neq |M| \end{cases}.$$

And

$$\sum_{M \in \mathsf{MI}(K)} \mathbf{S}_\alpha(\mathbf{v}_M) = \mathbf{MIV}_{\mathsf{R}}(K, \alpha).$$

So

$$
\begin{aligned}
&\mathbf{S}_{\alpha}^{\mathbf{I}_{\mathsf{R}}}(K)\\
&= \sum_{C \subseteq N} \frac{(c-1)!(n-c)!}{n!}(\mathbf{I}_{\mathsf{R}}(C) - \mathbf{I}_{\mathsf{R}}(C \setminus \{\alpha\}))\\
&= \sum_{C \subseteq N} \frac{(c-1)!(n-c)!}{n!}(\sum_{M \in \mathsf{MI}(K)} \mathbf{v}_M(C) - \sum_{M \in \mathsf{MI}(K)} \mathbf{v}_M(C \setminus \{\alpha\}))\\
&= \sum_{C \subseteq N} \frac{(c-1)!(n-c)!}{n!}(\sum_{M \in \mathsf{MI}(K)} (\mathbf{v}_M(C) - \mathbf{v}_M(C \setminus \{\alpha\})))\\
&= \sum_{C \subseteq N} (\sum_{M \in \mathsf{MI}(K)} \frac{(c-1)!(n-c)!}{n!}(\mathbf{v}_M(C) - \mathbf{v}_M(C \setminus \{\alpha\})))\\
&= \sum_{M \in \mathsf{MI}(K)} (\sum_{C \subseteq N} \frac{(c-1)!(n-c)!}{n!}(\mathbf{v}_M(C) - \mathbf{v}_M(C \setminus \{\alpha\})))\\
&= \sum_{M \in \mathsf{MI}(K)} \mathbf{S}_{\alpha}(\mathbf{v}_M)\\
&= \mathbf{MIV}_{\mathsf{R}}(K, \alpha).
\end{aligned}
$$

Note that this proof is similar to the corresponding proposition in (Hunter et al, 2008).

The Proof of Proposition 4.9

proof: Suppose that $\mathbf{S}_{\alpha}^{\mathbf{I}}$ is an inconsistency value satisfies Distribution, Symmetry, Minimality, Revised Decomposability and Revised MinInc. Then

- $\sum_{\alpha \in K} \mathbf{S}_{\alpha}^{\mathbf{I}}(K) = \mathbf{I}(K)$ (Distribution)
- $\mathbf{S}_{\alpha}^{\mathbf{I}}(K) = \mathbf{0}$ if α is a free formula in K. (Minimality)
- $\mathbf{S}_{\alpha}^{\mathbf{I}}(K) = \sum_{M \in \mathsf{MI}(K)} \mathbf{S}_{\alpha}^{\mathbf{I}}(M)$. (Revised Decomposability)

Then

$$\mathbf{S}_{\alpha}^{\mathbf{I}}(K) = \sum_{M \in \mathsf{MI}(K) \ s.t. \ \alpha \in M,} \mathbf{S}_{\alpha}^{\mathbf{I}}(M)$$

By symmetry,

$$\forall \alpha, \beta \in M, \quad \mathbf{S}_{\alpha}^{\mathbf{I}}(M) = \mathbf{S}_{\beta}^{\mathbf{I}}(M)$$

Then

$$\forall \alpha \in M, \quad \mathbf{S}_{\alpha}^{\mathbf{I}}(M) = \frac{1}{|M|} \cdot \mathbf{I}(M).$$

So,

$$\mathbf{S}_{\alpha}^{\mathbf{I}}(K) = \sum_{M \in \mathsf{MI}(K) \ s.t. \ \alpha \in M,} \frac{1}{|M|} \cdot \mathbf{I}(M)$$

By Revised MinInc, We know that if $M \in \mathsf{MI}(K)$, then $\mathbf{I}(M) = \mathbf{I}_{\mathsf{R}}(M)$.

$$\begin{aligned}
\mathbf{S}^{\mathbf{I}}_{\alpha}(K) &= \sum_{M \in \mathsf{MI}(K)\ s.t.\ \alpha \in M,} \frac{1}{|M|} \cdot \mathbf{I}_{\mathsf{R}}(M) \\
&= \sum_{M \in \mathsf{MI}(K)} \mathbf{MIV}_{\mathsf{R}}(M, \alpha) \\
&= \mathbf{MIV}_{\mathsf{R}}(K, \alpha) \\
&= \mathbf{S}^{\mathbf{I}_{\mathsf{R}}}_{\alpha}(K)
\end{aligned}$$

The Proof of Proposition 5.7

Proof: Suppose that S^{I}_{α} is an inconsistency value satisfies Distribution, Symmetry, Minimality, Revised Decomposability and weighted MinInc. Then

- $\sum_{\alpha \in K} S^{\mathsf{I}}_{\alpha}(K) = \mathsf{I}(K)$ (Distribution)
- $S^{\mathsf{I}}_{\alpha}(K) = 0$ if α is a free formula in K. (Minimality)
- $S^{\mathsf{I}}_{\alpha}(K) = \sum_{M \in \mathsf{MI}(K)} S^{\mathsf{I}}_{\alpha}(M)$. (Revised Decomposability)

Then

$$S^{\mathsf{I}}_{\alpha}(K) = \sum_{M \in \mathsf{MI}(K)\ s.t.\ \alpha \in M,} S^{\mathsf{I}}_{\alpha}(M)$$

By symmetry,

$$\forall \alpha, \beta \in M, \quad S^{\mathsf{I}}_{\alpha}(M) = S^{\mathsf{I}}_{\beta}(M)$$

Then

$$\forall \alpha \in M, \quad S^{\mathsf{I}}_{\alpha}(M) = \frac{1}{|M|} \cdot \mathsf{I}(M).$$

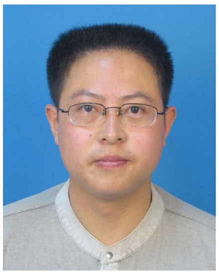
So,

$$S^{\mathsf{I}}_{\alpha}(K) = \sum_{M \in \mathsf{MI}(K)\ s.t.\ \alpha \in M,} \frac{1}{|M|} \cdot \mathsf{I}(M)$$

By Weighted MinInc, We know that if $M \in \mathsf{MI}(K)$, then $\mathsf{I}(M) = \mathsf{I}_{\mathsf{W}}(M) = w_{|M|}$.

$$\begin{aligned}
S^{\mathsf{I}}_{\alpha}(K) &= \sum_{M \in \mathsf{MI}(K)\ s.t.\ \alpha \in M,} \frac{w_{|M|}}{|M|} \\
&= \sum_{M \in \mathsf{MI}(K)} \mathsf{MIV}_{\mathsf{W}}(M, \alpha) \\
&= \mathsf{MIV}_{\mathsf{W}}(K, \alpha) \\
&= S^{\mathsf{I}_{\mathsf{W}}}_{\alpha}(K)
\end{aligned}$$

Author Biographies



Kedian Mu received the B.Sc. Degree in Applied Mathematics from Beijing Institute of Technology, Beijing, China, in 1997, the M.Sc. Degree in Probability and Mathematical Statistics from Beijing Institute of Technology, Beijing, China, in 2000, and the Ph.D. in Applied Mathematics from Peking University, Beijing, China, in 2003. From 2003 to 2005, He was a postdoctoral researcher at Institute of Computing Technology, Chinese Academy of Sciences, China. He is currently an Assistant Professor at School of Mathematical Sciences, Peking University, Beijing, China. His research interests include uncertain reasoning in artificial intelligence, knowledge engineering and science, and requirements engineering.

Weiru Liu is a Reader at the School of Electronics, Electrical Engineering and Computer Science, Queen's University Belfast. She received her B.Sc. and M.Sc. degrees in Computer Science from Jilin University, P.R. China, and her Ph.D. degree in Artificial Intelligence from the University of Edinburgh. Her main research interests include reasoning under uncertainty, uncertain and inconsistent knowledge and information fusion, belief and epistemic revision, and knowledge discovery in databases. She has published over 100 journal and conference papers in these areas.

Zhi Jin was awarded a B.Sc. in Computer Science from Zhejiang University, Hangzhou, China, in 1984, and studied for her M.Sc. in Computer Science (Expert System) and her Ph.D. in Computer Science (Artificial Intelligence) at National Defence University of Technology, Changsha, China. She was awarded a PhD in 1992. She is a senior member of China Computer Federation. She is currently a professor at School of Electronics Engineering and Computer Science, Peking University. Her research interests include Knowledge-based Systems, Artificial Intelligence, Requirements Engineering, Ontology Engineering, etc. Her current research focuses on Ontology-based Requirements Elicitation and Analysis. She has got about 70 publications, including co-authoring one book.

Correspondence and offprint requests to: Kedian Mu, School of Mathematical Sciences, Peking University, Beijing 100871, P.R.China. Email: mukedian@math.pku.edu.cn