
Measuring the blame of each formula for inconsistent prioritized knowledge bases

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Abstract

It is increasingly recognized that identifying the degree of blame or responsibility of each formula for inconsistency of a knowledge base (i.e. a set of formulas) is useful for making rational decisions to resolve inconsistency in that knowledge base. Most current techniques for measuring the blame of each formula with regard to an inconsistent knowledge base focus on classical knowledge bases only. Proposals for measuring the blames of formulas with regard to an inconsistent prioritized knowledge base have not yet been given much consideration. However, the notion of priority is important in inconsistency-tolerant reasoning. This article investigates this issue and presents a family of measurements for the degree of blame of each formula in an inconsistent prioritized knowledge base by using the minimal inconsistent subsets of that knowledge base. First of all, we present a set of intuitive postulates as general criteria to characterize rational measurements for the blames of formulas of an inconsistent prioritized knowledge base. Then we present a family of measurements for the blame of each formula in an inconsistent prioritized knowledge base under the guidance of the principle of proportionality, one of the intuitive postulates. We also demonstrate that each of these measurements possesses the properties that it ought to have. Finally, we use a simple but explanatory example in requirements engineering to illustrate the application of these measurements. Compared to the related works, the postulates presented in this article consider the special characteristics of minimal inconsistent subsets as well as the priority levels of formulas. This makes them more appropriate to characterizing the inconsistency measures defined from minimal inconsistent subsets for prioritized knowledge bases as well as classical knowledge bases. Correspondingly, the measures guided by these postulates can intuitively capture the inconsistency for prioritized knowledge bases.

Keywords: Inconsistency measures, prioritized knowledge base, minimal inconsistent subsets, blame of each formula in inconsistency.

1 Introduction

Many (if not all) real-world applications are characterized by the presence of inconsistency. Research in inconsistency handling has attracted significant attention in recent years because of the necessity for dealing with inconsistency in applications. Increasingly, it is recognized that measuring inconsistency is a crucial part of inconsistency handling in computer science as well as in artificial intelligence, and their applications [2].

A growing number of techniques for measuring inconsistency have been explored under different scenarios, such as knowledge merging [24], ontology management [23], software engineering and requirements negotiation [3, 19, 20], as well as in artificial intelligence research in general [2, 7–9, 11, 12, 14–16]. [10] provides a good review on some recent proposals for measuring inconsistency.

The overwhelming majority of the current proposals for measuring inconsistency are concentrated on the degree of inconsistency (or the amount of contradiction) for a whole knowledge base. In contrast, there are relatively few techniques for measuring the degree of blame or responsibility of each individual formula of a knowledge base for the inconsistency in that knowledge base. However, in many applications, it is desirable to resolve inconsistency by measuring and identifying the responsibility of each formula for that inconsistency. For example, in requirements engineering, developers or requirement analysts need to know the responsibility of each requirement involved in inconsistency to take an appropriate action for resolving the inconsistency.

The current proposals for measuring the blame of each formula for inconsistency of a knowledge base can be classified into two categories. The first category focuses on the distribution of an existing (syntax-based or model-based) measure of the degree of inconsistency of a knowledge base among formulas belonging to that knowledge base by using a cooperative game theory model [11]. The proportion of the existing measure distributed to an individual formula of a knowledge base is considered as a measurement of the blame of that formula for the inconsistency of that knowledge base. The Shapley inconsistency value presented by Hunter and Konieczny [11] is the most representative work of the first category, in which the Shapley value [1, 26], the well-known coalitional game model, has been used to distribute an existing measure of the overall inconsistency of a knowledge base to each formula belonging to that knowledge base. In contrast, the second category such as [12] argues that it is natural and intuitive to define syntactic measures of the blame of each formula of a knowledge base for its inconsistency from minimal inconsistent subsets of the knowledge base directly. Note that it follows the viewpoint that minimal inconsistent subsets of a knowledge base could be considered as the purest form of inconsistency of the knowledge base [25]. Moreover, Hunter and Konieczny argued that syntax sensitive measures are necessary for some applications such as requirements engineering [12]. The MinInc inconsistency values presented in [12] and the scoring function of singleton sets presented in [13] can be considered as the most representative works of the second category. In particular, an interesting link between the two approaches is that the MinInc inconsistency value MIV_C is also a special Shapley inconsistency value [12].

Note that approaches in both the first and the second categories are concerned with classical knowledge bases only. Proposals for measuring the blame of each formula belonging to a prioritized knowledge base for the inconsistency of that base have not yet been given deserved attention. That is, the current proposals for measuring the blames of formulas for inconsistency of a knowledge base do not take the priority of each formula into account. However, as a description of the importance or reliability of the corresponding knowledge, the priority of each formula always plays an important role in inconsistency handling in artificial intelligence and many applications, such as belief revision [4], knowledge base merging [6, 24] and inconsistency handling in requirements engineering [20, 21]. It is natural to take the priority of each formula into account when identifying the blames of formulas for inconsistency of a prioritized knowledge base.

The Shapley value model of the first category is not appropriate for characterizing the blame of each prioritized formula for the inconsistency of a prioritized knowledge base, even if we consider the priority of each formula when measuring the amount of inconsistency of that prioritized knowledge base. To illustrate this, let us consider a scenario of two conflicting witnesses. Suppose that Alice and Bob are witnesses about the case of John being accused

of breaking the speeding limit in an traffic accident. Alice claims that John was breaking the speeding limit when the traffic accident happened. But Bob claims that John wasn't over the speeding limit when the accident happened. Then we have an inconsistent knowledge base from the two witnesses, i.e. $K = \{\text{Break}(\text{John}, \text{speed_limit}), \neg \text{Break}(\text{John}, \text{speed_limit})\}$. If we know that Alice has a good reputation, but Bob is a drunken witness, we may consider that $\text{Break}(\text{John}, \text{speed_limit})$ is much more reliable than $\neg \text{Break}(\text{John}, \text{speed_limit})$. Intuitively, the degree of responsibility of $\neg \text{Break}(\text{John}, \text{speed_limit})$ for the inconsistency in K should be more than that of $\text{Break}(\text{John}, \text{speed_limit})$. However, the Shapley inconsistency value cannot make a distinction between the blames of $\text{Break}(\text{John}, \text{speed_limit})$ and $\neg \text{Break}(\text{John}, \text{speed_limit})$, since it assigns half of the amount of inconsistency in K to $\text{Break}(\text{John}, \text{speed_limit})$ and $\neg \text{Break}(\text{John}, \text{speed_limit})$, respectively.

As mentioned earlier, the fundamental viewpoint of the second category is that minimal inconsistent subsets of a (classical or prioritized) knowledge base can be considered as the purest form of inconsistency in that base. Following this viewpoint in the case of prioritized knowledge bases, it is also natural to explore connections between the measures for the blame of each prioritized formula for the inconsistency of the knowledge base the formula belongs to and the minimal inconsistent subsets of that base. Our previous article [19] can be considered as an attempt to follow this viewpoint for prioritized knowledge bases, in which we explored a possible combination of the scoring function presented in [13] and the priority levels of formulas. However, as pointed out in [12], the scoring function of a singleton set for a given knowledge base is very sketchy for measuring the degree of blame of the corresponding formula for the inconsistency of the knowledge base. With regard to another representative proposal of the second category, one of the fundamental characteristics of the MinInc inconsistency value MinInc and some revised MinInc inconsistency values presented in [22] is that each formula of a given minimal inconsistent knowledge base M takes the same degree of responsibility (i.e. $\frac{1}{|M|}$) for the inconsistency of M . This is not always satisfiable for prioritized knowledge bases such as the example of two conflicting witnesses mentioned above.

In this article, we argue that it is desirable to have proportional distributions of the overall inconsistency of a minimal inconsistent prioritized knowledge base among formulas belonging to that base, according to severity of contribution of each formula to cause the inconsistency. That is, a reasonable measurement of the blame of each formula of a prioritized knowledge base for the inconsistency of that base should obey the so-called *principle of proportionality*. In order to rationally justify any measures based on this principle, we first propose a set of properties to characterize measures of the blames of formulas in the inconsistency of a prioritized knowledge base, which accords with the principle as well as the view of minimal inconsistent subsets being the purest form of inconsistency. In particular, these properties take the special characteristics of minimal inconsistent subsets as well as the priorities of formulas into account. This makes the set of properties more appropriate to capturing the most intuitive constraints in general such as most of properties presented in [11] as well as special constraints presented in this article for inconsistency measures based on minimal inconsistent subsets. We then define a family of measures of the blame of each formula for the inconsistency of a prioritized knowledge base under the guidance of the principle of proportionality. We finally show these measures satisfy these intuitive properties. Compared to the current proposals, the measures guided by these properties are more succinct and intuitive to capture the degree of blame of each formula for inconsistency in prioritized knowledge bases as well as in classical knowledge bases. Especially, the corresponding measures in the case of classical knowledge bases provide support for the intuition about the inconsistency of minimal inconsistent subsets illustrated by the well known Lottery Paradox [15].

The rest of this article is organized as follows. Some preliminaries are given in the next Section. In Section 3, we provide some rational properties to characterize an intuitive and natural measure of the blames for formulas. In Section 4, we define a family of natural measures of the blames of formulas from minimal inconsistent subsets. In Section 5, we present a case study to illustrate the application of our approach to requirements engineering. In Section 6, we compare our work with related works. Finally, we conclude this article in Section 7.

2 Preliminaries

Throughout this article, we will only consider a finite propositional language. Let \mathcal{P} be a finite set of propositional symbols and \mathcal{L} a propositional language built from \mathcal{P} under connectives $\{\neg, \wedge, \vee, \rightarrow\}$. We use a, b, c, \dots to denote the propositional symbols, and $\alpha, \beta, \gamma, \dots$ to denote the propositional formulas.

A classical (or flat) knowledge base \tilde{K} is a finite set of propositional formulas. We use $\tilde{\mathcal{K}}_{\mathcal{L}}$ to denote the set of classical knowledge bases definable from the language \mathcal{L} .

Compared to classical knowledge bases, prioritized knowledge bases take the priority level of each formula into account. That is, each formula in a prioritized knowledge base is attached with a priority level. In different applications, a priority level has different meanings. For example, the priority of a requirement statement in Requirements Engineering characterizes how important the requirement is for a computer system, while the priority level of a piece of belief describes how reliable this piece of belief is. In this article, we interpret that the priority level of a formula in a given knowledge base embodies how *preferred* the formula is with regard to other formulas in the knowledge base.

DEFINITION 2.1 (Prioritized knowledge bases)

A prioritized knowledge base K is a finite set of propositional formulas, each being attached with its priority level, i.e.

$$K = \{(\alpha_1, P(\alpha_1)), \dots, (\alpha_n, P(\alpha_n))\},$$

where P is a prioritization function from \mathcal{L} to a set of priorities Pri .

For simplicity, we use α^P to denote a prioritized formula $(\alpha, P(\alpha))$. Then a prioritized knowledge base K can be represented as $K = \{\alpha_1^P, \dots, \alpha_n^P\}$.

This definition of prioritized knowledge base shares the spirit of the definition of prioritized observation base in [4] in which each formula is attached with a integer value indicating its priority.

Generally, there are two kinds of priorities used in most application domains, i.e. numerical priorities and qualitative priorities.

- The first kind of priorities Pri_I is a set of numerical valuations (typically, $[0, 1]$, \mathbf{R}^+ , etc). The corresponding prioritization function $P_I : \mathcal{L} \mapsto \text{Pri}_I$ is a function that assigns a precise numerical valuation to each formula such that the bigger the priority valuation of a formula, the more preferred the formula is. For example, the necessity degree attached with each formula in possibilistic knowledge base [6] may be considered as a priority of the first kind, in which Pri_I is $[0, 1]$. For convenience and simplicity and without losing generality, we assume that Pri_I is $[0, 1]$ in the rest of the article. As mentioned above, the same value in $[0, 1]$ may have different meanings in different applications. Therefore, the meaning of the most preferred level (e.g. 1) and that of the lowest priority value (e.g. 0) depend on individual applications. For example, in possibilistic logic, $(\alpha, 1)$ is interpreted as α being definitely true.
- In contrast, the second kind of priorities Pri_{II} consists of finite ordered qualitative priority levels, such as the 3-level priority set $\{High, Medium, Low\}$ used in requirements engineering [19, 21].

The corresponding prioritization function $P_{\Pi} : \mathcal{L} \mapsto \text{Pri}_{\Pi}$ assigns a qualitative level such as *High* or *Low* instead of a precise numerical valuation to each propositional formula. As such, the propositional formulas were grouped into $|\text{Pri}_{\Pi}|$ categories. More generally, if we use a natural number i to denote the i -th priority level, then the corresponding prioritization function can be restated as $P'_{\Pi} : \mathcal{L} \mapsto \{1, 2, \dots, |\text{Pri}_{\Pi}|\}$ such as any formula with a smaller value is more preferred than formulas with a larger value. In particular, such a prioritized knowledge base

$$K = \{(\alpha_1, P'_{\Pi}(\alpha_1)), \dots, (\alpha_n, P'_{\Pi}(\alpha_n))\}$$

can be alternatively expressed as a $|\text{Pri}_{\Pi}|$ -tuple of sets of formulas, i.e.

$$\langle \{\alpha_i | \alpha_i^P \in K, P'_{\Pi}(\alpha_i) = 1\}, \dots, \{\alpha_i | \alpha_i^P \in K, P'_{\Pi}(\alpha_i) = |\text{Pri}_{\Pi}|\} \rangle$$

Especially, \emptyset can also be reexpressed as $\langle \emptyset, \dots, \emptyset \rangle$. Obviously, for each k , $\{\alpha_i | \alpha_i^P \in K, P'_{\Pi}(\alpha_i) = k\}$ is a set of all formulas at k -th priority level of K . From now on, we call this set of formulas (possibly empty) the k -th stratum of K . Note again that if we interpret the k -th stratum of K as the set of formulas of reliability level k , then K can be considered as a special prioritized observations base defined in [4].

Generally, the relative priority of a prioritized knowledge base should depend on priority levels of formulas belonging to that knowledge base. However, for qualitative prioritization, it is often hard to use a single value to capture the relative priority of a knowledge base consisting of qualitatively prioritized formulas, since it is very subtle to replace each qualitative level with a numerical weight [21]. Correspondingly, the operator of comparison and integration should take the stratified structure of such a knowledge base into account. To address this, we need two different approaches to handling these two types of prioritized knowledge bases.

To distinguish these two types of prioritizes assigned to formulas, we call a prioritized knowledge base a Type-I prioritized knowledge base if it adopts the first kind of prioritization function. Otherwise, we call it a Type-II prioritized knowledge base. For example, a knowledge base in possibilistic logic can be considered as a Type-I prioritized knowledge base, in which $(\alpha, P_I(\alpha))$ is interpreted as the degree of necessity of α is at least $P_I(\alpha)$. To make a distinction, we use \hat{K} to denote a Type-I prioritized knowledge base. Also, for convenience, we assume that the priority of each formula in a Type-I prioritized knowledge base is non-zero. In contrast, we use K or $\langle K(1), \dots, K(m) \rangle$ to denote a Type-II prioritized knowledge base, in which $K(i)$ is the i -th stratum of K for each $1 \leq i \leq m$. Note that $\alpha^P \in K$ if and only if $\alpha \in K(P'_{\Pi}(\alpha))$. So, $K' = \langle K'(1), \dots, K'(m) \rangle$ is a subset of K if and only if $K'(i) \subseteq K(i)$ for each i . Moreover, we abuse the notation and write α instead of α^P for a prioritized formula $\alpha^P \in K$. We must point out that for Type-II prioritized knowledge bases, we assume that there is no explicit or implicit numerical relation between these priority levels, except ordering relation. So, the stratified structure is essential to describe the Type-II knowledge base. For example, we may use $\{1, 2, 3\}$ as the scale of priority levels. But we cannot consider the degree of preference of α at level 1 is 3 times to that of β at level 3. It is the main difference between these two types knowledge bases.

The classical knowledge base associated with a prioritized knowledge base K , denoted K^* , is defined as $K^* = \{\alpha | (\alpha, P(\alpha)) \in K\}$. On the other hand, a classical knowledge base $\tilde{K} = \{\alpha_1, \dots, \alpha_n\}$ corresponds to a Type-I prioritized knowledge base $\hat{K} = \{(\alpha_1, 1), \dots, (\alpha_n, 1)\}$, or a Type-II prioritized knowledge base $K = \langle \tilde{K} \rangle$. Then we can take each classical knowledge base \tilde{K} as a special kind of prioritized knowledge base (i.e. \hat{K} or K), and we simply use K to denote either a prioritized or a flat knowledge base if there is no confusion. Moreover, we use $\mathcal{K}_{\mathcal{L}}$ (respectively $\hat{\mathcal{K}}_{\mathcal{L}}$) and $\mathcal{M}_{\mathcal{L}}$

(respectively $\widehat{\mathcal{M}}_{\mathcal{L}}$) to denote the set of (respectively Type-I) prioritized knowledge bases and minimal inconsistent prioritized knowledge bases definable from formulas of the language \mathcal{L} , respectively. Evidently, $\mathcal{M}_{\mathcal{L}} \subset \mathcal{K}_{\mathcal{L}}$.

A classical knowledge base \tilde{K} is inconsistent if there is a formula α such that $\tilde{K} \vdash \alpha$ and $\tilde{K} \vdash \neg \alpha$. We abbreviate $\alpha \wedge \neg \alpha$ as \perp if there is no confusion. Then an inconsistent \tilde{K} is denoted by $\tilde{K} \vdash \perp$. A prioritized knowledge base K is inconsistent if K^* is inconsistent.

Moreover, an inconsistent (prioritized or classical) knowledge base K is called a *minimal inconsistent set* if none of its proper subsets is inconsistent. If $K' \subseteq K$ and K' is a minimal inconsistent set, then we call K' a *minimal inconsistent subset* of K .

Let $\text{MI}(K)$ be the set of all the minimal inconsistent subsets of K , i.e.

$$\text{MI}(K) = \{K' \subseteq K \mid K' \vdash \perp \text{ and } \forall K'' \subset K', K'' \not\vdash \perp\}.$$

The minimal inconsistent subsets can be considered as the purest form of inconsistency for conflict resolution where the syntactic representation of the information is important, since removing one formula from each minimal inconsistent subset would be sufficient to resolve the inconsistency [25]. In contrast, a *free formula* of a knowledge base K is referred to as a formula of K that does not belong to any minimal inconsistent subset of K . In this article, we use $\text{FREE}(K)$ to denote the set of free formulas of K .

EXAMPLE 2.1

Consider $K_1 = \langle \{a, \neg b\}, \{c\}, \{\neg a, b, d\} \rangle$. Evidently, K_1 is inconsistent. The set of minimal inconsistent subsets of K_1 is $\text{MI}(K_1) = \{M_1, M_2\}$, where

$$M_1 = \langle \{a\}, \emptyset, \{\neg a\} \rangle, \quad M_2 = \langle \{\neg b\}, \emptyset, \{b\} \rangle.$$

In contrast, both c and d are free formulas of K_1 , i.e. $\text{FREE}(K_1) = \langle \emptyset, \{c\}, \{d\} \rangle$.

Let f and g be two functions of knowledge bases and minimal inconsistent knowledge bases, respectively. From now on, just for simplicity for discussion, we abbreviate

$$\forall K, f(K) = \begin{cases} \sum_{M \in \text{MI}(K)} g(M), & \text{if } \text{MI}(K) \neq \emptyset, \\ 0, & \text{if } \text{MI}(K) = \emptyset. \end{cases}$$

as

$$\forall K, f(K) = \sum_{M \in \text{MI}(K)} g(M).$$

Hunter *et al.* have argued that it is natural to explore links between measuring inconsistency for a knowledge base and the minimal inconsistent subsets of that base in [12, 13]. They defined the MI inconsistency measure for the amount of inconsistency for a classical knowledge base as well as the MinInc Inconsistency Value for the blame of each formula for the inconsistency of a classical knowledge base from minimal inconsistent subsets of that base [12].

DEFINITION 2.2 (The MI inconsistency measure [12])

The MI inconsistency measure is defined as the number of minimal inconsistent subsets of a classical knowledge base \tilde{K} , i.e.:

$$I_{\text{MI}}(\tilde{K}) = |\text{MI}(\tilde{K})|.$$

DEFINITION 2.3 (The MinInc Inconsistency Value MIV_C [12])

The MinInc Inconsistency Value MIV_C is defined as follows:

$$\text{MIV}_C(\tilde{K}, \alpha) = \sum_{\tilde{M} \in \text{MI}(\tilde{K}) \text{ s.t. } \alpha \in \tilde{M}} \frac{1}{|\tilde{M}|}$$

for any classical knowledge base K and propositional formula α .

The MinInc Inconsistency Value $\text{MIV}_C(\tilde{K}, \alpha)$ gives the blame of α for the inconsistency of \tilde{K} . It is a particular type of Shapley inconsistency value defined by I_{MI} , and can be completely characterized in terms of five axioms defined in [12]. In particular, the MinInc axiom, one of the five axioms, states that each minimal inconsistent subset brings the same amount of conflict:

- If $\tilde{M} \in \text{MI}(\tilde{K})$, then $\text{I}_{\text{MI}}(\tilde{M}) = 1$.

Evidently, the combination of MinInc axiom and the definition of MIV_C signifies that the amount of inconsistency of a minimal inconsistent knowledge base is *shared equally* among all the formulas belonging to this base.

Some properties were also presented to characterize a basic inconsistency measure for classical knowledge bases [11, 12].

DEFINITION 2.4 (Basic Inconsistency Measure [12])

An inconsistency measure I is called a basic inconsistency measure if it satisfies the following properties: $\forall \tilde{K}, \tilde{K}' \in \tilde{\mathcal{K}}_{\mathcal{L}}, \forall \alpha, \beta \in \mathcal{L}$:

- **Consistency:** $\text{I}(\tilde{K}) = 0$ if \tilde{K} is consistent.
- **Monotony:** $\text{I}(\tilde{K} \cup \tilde{K}') \geq \text{I}(\tilde{K})$.
- **Free Formula Independence:** If α is a free formula of $\tilde{K} \cup \{\alpha\}$, then $\text{I}(\tilde{K} \cup \{\alpha\}) = \text{I}(\tilde{K})$.
- **Dominance:** If $\alpha \vdash \beta$ and $\alpha \not\vdash \perp$, then $\text{I}(\tilde{K} \cup \{\alpha\}) \geq \text{I}(\tilde{K} \cup \{\beta\})$.

The consistency property requires that a desirable inconsistency measure assigns null to a consistency base. The monotony property states that as the size of a knowledge base increases, the amount of inconsistency cannot decrease. The free formula independence property requires that adding or deleting a free formula cannot change the amount of inconsistency of the base. As explained in [11], the dominance property states that logically stronger formulas bring (potentially) more conflicts. However, the MI inconsistency measure I_{MI} , considered as a basic inconsistency measure in [12], does not satisfy the dominance property. To illustrate this, consider $\tilde{K} = \{a, a \wedge c, \neg b\}$. Suppose that $\alpha = a \wedge c \wedge (\neg a \vee b)$ and $\beta = \neg a \vee b$. Then $\text{MI}(\tilde{K} \cup \{\beta\}) = \{\{a, \neg a \vee b, \neg b\}, \{a \wedge c, \neg a \vee b, \neg b\}\}$ and $\text{MI}(\tilde{K} \cup \{\alpha\}) = \{a \wedge c \wedge (\neg a \vee b), \neg b\}$. So,

$$\text{I}_{\text{MI}}(\tilde{K} \cup \{\alpha\}) = 1 < \text{I}_{\text{MI}}(\tilde{K} \cup \{\beta\}) = 2.$$

Because the minimal inconsistent subset is syntax sensitive, if we replace β with α , some inconsistencies in the form of distinctive minimal inconsistent subsets of $\tilde{K} \cup \{\beta\}$ will be expressed by either the same minimal inconsistent subset in which β is replaced by α , or by a minimal inconsistent subset with a smaller size of $\tilde{K} \cup \{\alpha\}$, as illustrated by the example above. If we consider the amount of inconsistency of a knowledge base as the sum of the amounts of inconsistency of all minimal inconsistent subsets, to satisfy the dominance property, the following condition must hold:

- for any two minimal inconsistent knowledge bases \tilde{M}_1 and \tilde{M}_2 , $\text{I}(\tilde{M}_1) \geq k \cdot \text{I}(\tilde{M}_2)$ for any k if $|\tilde{M}_1| < |\tilde{M}_2|$.

It is difficult to satisfy this condition if we use a single value to measure a knowledge base. To the best of our knowledge, the MI inconsistency vectorial measurer presented in [22] satisfies the dominance property, though an inconsistency measure satisfies the properties presented in [22] does not always satisfy the dominance property. On the other hand, the dominance property is inappropriate to characterize the inconsistency measure for prioritized knowledge bases. To illustrate this, consider $K = \langle \{a\}, \{\neg a\}, \emptyset \rangle$ and $K' = \langle \emptyset, \{\neg a\}, \{b \wedge (b \rightarrow a)\} \rangle$. Intuitively, K is more inconsistent than K' since a is more important than $b \wedge (b \rightarrow a)$, although $b \wedge (b \rightarrow a) \vdash a$. To address these, we do not consider the dominance property as a mandatory property in this article.

3 Properties for characterizing inconsistency measures from minimal inconsistent subsets

Contrary to the current proposals for discussing or analyzing the properties of a given measure for the blames of formulas for the inconsistency in a knowledge base, in this section, first of all, we discuss the intuitive properties for characterizing inconsistency measures for a prioritized knowledge base defined from minimal inconsistent subsets of that base.

Recall the properties of a basic inconsistency measure for classical knowledge bases presented in [11, 12], none of the four intuitive properties addresses the special characteristics of measures defined from minimal inconsistent subsets explicitly. That is, these four properties describe some common characteristics of syntactic inconsistency measures, but they are not specific enough to characterize the inconsistency measures for a knowledge base defined from its minimal inconsistent subsets. To illustrate, consider the MinInc axiom presented in [12], it states that each minimal inconsistent classical knowledge base has the same amount of conflict. Although it does not contradict the other properties, it contradicts the intuition illustrated by the lottery paradox [12, 15], which states that the bigger the size of a minimal inconsistent set, the smaller the amount of inconsistency it has.

On the other hand, compared to measures for the classical knowledge bases, measures for the inconsistency of prioritized knowledge bases should take the priority of each formula into account. That is, we need to consider the strength of inconsistency of each minimal inconsistent subset as well as the degree of inconsistency. However, the strength of inconsistency of a minimal inconsistent subset should take into account the significance (i.e. globally relative priority) of that subset, which is induced by the priority of each formula belonging to that subset. In other words, to measure the amount of inconsistency of minimal inconsistent prioritized knowledge base, we need to measure the significance of each minimal inconsistent subset based on the priority of each formula belonging to that base as the starting point. Intuitively, a measure for the significance of a prioritized knowledge base K , denoted as $\text{Sig}(K)$, should be subject to the following assumptions:

- (S1) Zero Significance: $\text{Sig}(K) = 0$ iff $K = \emptyset$.
- (S2) Preference: $\text{Sig}(K \cup \{\alpha^P\}) \geq \text{Sig}(K \cup \{\beta^P\})$ if $\alpha^P, \beta^P \notin K$ and α is more preferred than β .
- (S3) Monotony: $\forall K, K' \in \mathcal{K}_{\mathcal{L}}, \text{Sig}(K \cup K') \geq \text{Sig}(K)$.
- (S4) Upper Bound: $\forall K \in \mathcal{K}_{\mathcal{L}}, \text{Sig}(K) \leq \text{Sig}(K^*)$.

The assumption of Zero Significance states that an empty knowledge base has null significance. The assumption of Preference states that more preferred formulas bring more significance. The Monotony assumption states that as the size of a knowledge base increases, the significance of the base should not be decreased. The Upper Bound assumption requires that any reasonable measure of significance of a prioritized knowledge base cannot be greater than that of the classical knowledge base associated with that base, since each formula of the classical knowledge base is not less preferred than the

corresponding formula of the prioritized knowledge base. Roughly speaking, the upper bound states that the classical knowledge base is the most preferred among all the prioritized knowledge bases with the same classical formulas but different priorities.

The current approaches to measuring inconsistency of a knowledge base based on minimal inconsistent subsets such as [12] and [13] consider the sum of the amount of inconsistency of all the minimal inconsistent subsets of a knowledge base as the amount of inconsistency of that base. To illustrate this, recall the MI inconsistency measure I_{MI} defined in [12], it considers the number of minimal inconsistent subsets of a classical knowledge base as a measure of the amount of inconsistency of that base, while the MinInc property states that each minimal inconsistent subset brings the same amount of conflict. In this article, we follow this viewpoint in the case of prioritized knowledge bases. Then the characterization of measures of the amount of inconsistency for prioritized knowledge bases should focus on the measures of the amount of inconsistency for the minimal inconsistent knowledge bases. Intuitively, a desirable measure for the amount of inconsistency for a prioritized knowledge base defined from minimal inconsistent subsets of that base, denoted Inc , should satisfy the following properties:

- (I1) MinInc Additivity: $\forall K \in \mathcal{K}_{\mathcal{L}}, Inc(K) = \sum_{M \in MI(K)} Inc(M)$.
- (I2) Contradiction: $\forall M \in \mathcal{M}_{\mathcal{L}}, Inc(M) > 0$.
- (I3) Monotony w.r.t. Significance: $\forall M_1, M_2 \in \mathcal{M}_{\mathcal{L}}$, such that $|M_1| = |M_2|$, if $Sig(M_1) \leq Sig(M_2)$, then $Inc(M_1) \leq Inc(M_2)$.
- (I4) Attenuation w.r.t. Size: $\forall M_1, M_2 \in \mathcal{M}_{\mathcal{L}}$ s.t. $\forall \alpha^P, \beta^P \in M_1 \cup M_2, P(\alpha) = P(\beta)$, if $|M_1| < |M_2|$, then $Inc(M_1) > Inc(M_2)$.
- (I5) Almost Consistency: $\lim_{|M| \rightarrow +\infty} Inc(M) = 0$ for $M \in \mathcal{M}_{\mathcal{L}}$.

Note that the MinInc Additivity property requires that the amount of inconsistency in a knowledge base is equal to the sum of the amounts of inconsistency in all the minimal inconsistent subsets of that base. The Contradiction property states that each minimal inconsistent knowledge base should contribute to a non-zero amount of contradiction. The Monotony w.r.t. Significance property states that as the significance of a minimal inconsistent knowledge base with a given size increases, the amount of inconsistency that it contributes cannot decrease. In contrast, the Attenuation w.r.t. Size property states that as the size of a minimal inconsistent knowledge base consisting of formulas with the same priority increases, the amount of inconsistency decreases. The Almost Consistency property states that as the size of a minimal inconsistent set increases, the amount of inconsistency that it contributes to gradually reduces to zero, i.e. the minimal inconsistent set becomes nearly consistent if its size is large enough. Note that the Attenuation w.r.t. Size property and the Almost consistency property are more intuitive than the MinInc property presented in [12] which states that each minimal inconsistent subset brings the same amount of conflict in the case of classical knowledge bases. To illustrate this, let us consider the lottery paradox which motivated Knight to propose his approach [15]. The lottery paradox presented in [17] considered an n -ticket lottery known to be fair and to have exactly one winner. It is rational to accept *for any individual ticket i of the lottery that ticket i will not win*, since the probability of ticket i being the winner cannot exceed a high enough threshold due to the fairness of the lottery. Then $K_n = \{\neg w_1, \dots, \neg w_n, w_1 \vee \dots \vee w_n\}$ is a minimal inconsistent knowledge base about the lottery, where for each i , w_i asserts that ticket i will win the lottery. Intuitively, if there are a sufficiently large number of tickets in the lottery, the knowledge base K_n is *nearly consistent*, whereas K_n is *highly inconsistent* if there are only two or three tickets. Evidently, the MinInc property conflicts with this intuition. In contrast, the Attenuation

w.r.t. Size property and the Almost Consistency property comply with the intuition that as the size of minimal inconsistent subset increases, its inconsistency becomes more tolerable [12, 15].

On the other hand, as shown by the following lemma, in the case of classical knowledge bases, (I1) and (I2) subsume the former three properties for a basic inconsistency measure presented in [12].

LEMMA 3.1

If Inc satisfies (I1) and (I2), then Inc satisfies the properties of Consistency, Monotony, and Free Formula Independence in the case of classical knowledge bases.

For readability, proofs for propositions, lemmas, and corollaries are given in the Appendix. Recall the MinInc inconsistency value defined to measure the blame of each formula for the inconsistency of a classical knowledge base, it essentially accords with the principle of equal share, i.e. the amount of inconsistency of a minimal inconsistent subset should be shared equally among all the formulas belonging to that subset. As illustrated earlier, the principle of equal share is inappropriate for characterizing the measures for the blame of each formula for the inconsistency of a prioritized knowledge base.

However, the principle of proportionality may be used to guide the characterization of the blame of formulas for inconsistency in a prioritized knowledge base. This general principle argues that a reasonable measurement of the blame of an individual formula for inconsistency in a minimal inconsistent prioritized knowledge base should be proportionate to the gravity of its conduct in causing the inconsistency in that knowledge base. To obey to the principle of proportionality, we need to measure the severity of contribution of each formula to inconsistency of a prioritized knowledge base first. We start with the definition of opposed formulas.

DEFINITION 3.1 (Opposed formulas)

Let M be a minimal inconsistent prioritized knowledge base and α^P a formula being attached with a priority level. Then the set of opposed formulas to α^P w.r.t. M , denoted $\text{Opp}(M, \alpha^P)$, is defined as

$$\text{Opp}(M, \alpha^P) = \begin{cases} \{\alpha^P\}, & \text{if } M = \{\alpha^P\}, \\ M - \{\alpha^P\}, & \text{if } \{\alpha^P\} \subset M, \\ \emptyset, & \text{if } \alpha^P \notin M. \end{cases}$$

The opposed formulas to an individual formula in a minimal inconsistent knowledge base are formulas that would be disengaged from inconsistency if that formula was removed from the base. In particular, for a singleton set $M = \{\alpha^P\}$, the opposed formula to α^P is α^P , since α is a self-contradictory formula. Intuitively, the relative importance of opposed formulas to a given formula may be considered as a measure of severity of contribution of that formula to inconsistency in a minimal inconsistent knowledge base.

EXAMPLE 3.1

Consider $K_2 = \langle \{b\}, \{\neg b, c\}, \{a \wedge \neg a\} \rangle$. Then the set of minimal inconsistent subsets of K_2 is $\text{MI}(K_2) = \{M_1, M_2\}$, where

$$M_1 = \langle \emptyset, \emptyset, \{a \wedge \neg a\} \rangle, \quad M_2 = \langle \{b\}, \{\neg b\}, \emptyset \rangle.$$

Then the set of opposed formulas to $a \wedge \neg a$ w.r.t. M_1 is as follows:

$$\text{Opp}(M_1, a \wedge \neg a) = M_1.$$

The set of opposed formulas to each formula of K_2 w.r.t M_2 is given as follows:

$$\begin{aligned}\text{Opp}(M_2, a \wedge \neg a) &= \langle \emptyset, \emptyset, \emptyset \rangle, & \text{Opp}(M_2, \neg b) &= \langle \{b\}, \emptyset, \emptyset \rangle. \\ \text{Opp}(M_2, b) &= \langle \emptyset, \{\neg b\}, \emptyset \rangle, & \text{Opp}(M_2, c) &= \langle \emptyset, \emptyset, \emptyset \rangle.\end{aligned}$$

Combining the principle of proportionality and the view of minimal inconsistent subsets as the purest form of inconsistency, a desirable measure for the blame of each formula belonging to a prioritized knowledge base for the inconsistency of that base, denoted as **Blame**, should satisfy the following properties:

- (D1) Accumulation: $\forall K \in \mathcal{K}_{\mathcal{L}}, \forall \alpha^P \in K, \text{Blame}(K, \alpha^P) = \sum_{M \in \text{MI}(K)} \text{Blame}(M, \alpha^P)$.
- (D2) Innocence: $\forall M \in \mathcal{M}_{\mathcal{L}}, \forall \alpha^P \notin M, \text{Blame}(M, \alpha^P) = 0$.
- (D3) Necessity: $\forall M \in \mathcal{M}_{\mathcal{L}}, \forall \alpha^P \in M, \text{Blame}(M, \alpha^P) > 0$.
- (D4) MinInc Distribution: $\forall M \in \mathcal{M}_{\mathcal{L}}, \sum_{\alpha^P \in M} \text{Blame}(M, \alpha^P) = \text{Inc}(M)$.
- (D5) Proportionality: Given $M \in \mathcal{M}_{\mathcal{L}}, \forall \alpha^P \in M$,

$$\text{Blame}(M, \alpha^P) = c(M) \times \text{Sig}(\text{Opp}(M, \alpha^P)),$$

where $c(M)$ is a constant with regard to M .

The Accumulation property requires that the blame of each formula for the inconsistency of a prioritized knowledge base is the accumulation of the blames of that formula for the inconsistency of each minimal inconsistent subset of that base. The Innocence property states that any formula not belonging to a minimal inconsistent prioritized knowledge base dismisses any responsibility for the inconsistency in that minimal inconsistent base. The Necessity property states that each formula belonging to a minimal inconsistent prioritized knowledge base must bear some responsibility for the inconsistency of that minimal inconsistent base. The MinInc Distribution property states that the blame of a minimal inconsistent prioritized knowledge base is shared among the formulas belonging to that minimal inconsistent base. The Proportionality property requires that the blame of each formula in a minimal inconsistent prioritized knowledge base is proportionate in severity to the gravity of the conduct of the formula in inconsistency of that minimal inconsistent knowledge base. Note that (D1)–(D4) comply with the view that the minimal inconsistent subsets of a knowledge base are the purest form of inconsistency of that base. (D5) requires that the degree of blame of each formula must accord with the principle of proportionality, i.e. the more significant the opposed formulas to the formula are, the more severe the deserved blame of the formula should be.

In the case of classical knowledge bases, the following properties for an expected measure **Blame** were enumerated in [11, 12]:

- Distribution: $\forall \tilde{K} \in \tilde{\mathcal{K}}_{\mathcal{L}}, \sum_{\alpha \in \tilde{K}} \text{Blame}(\tilde{K}, \alpha) = \text{Inc}(\tilde{K})$.
- Symmetry: If $\exists \alpha, \beta \in \tilde{K}$ s.t. for all $\tilde{K}' \subseteq \tilde{K}$ s.t. $\alpha, \beta \notin \tilde{K}'$, $\text{Inc}(\tilde{K}' \cup \{\alpha\}) = \text{Inc}(\tilde{K}' \cup \{\beta\})$, then $\text{Blame}(\tilde{K}, \alpha) = \text{Blame}(\tilde{K}, \beta)$.
- Minimality: If α is a free formula of \tilde{K} , then $\text{Blame}(\tilde{K}, \alpha) = 0$.
- Dominance: If $\alpha \vdash \beta$ and $\alpha \not\vdash \perp$, then $\text{Blame}(\tilde{K}, \alpha) \geq \text{Blame}(\tilde{K}, \beta)$.
- Decomposability: If $\text{MI}(\tilde{K} \cup \tilde{K}') = \text{MI}(\tilde{K}) \oplus \text{MI}(\tilde{K}')$, then

$$\text{Blame}(\tilde{K} \cup \tilde{K}', \alpha) = \text{Blame}(\tilde{K}, \alpha) + \text{Blame}(\tilde{K}', \alpha),$$

where $\text{MI}(\tilde{K}) \oplus \text{MI}(\tilde{K}')$ is referred to as $\text{MI}(\tilde{K}) \cup \text{MI}(\tilde{K}')$ for two mutually exclusive sets $\text{MI}(\tilde{K})$ and $\text{MI}(\tilde{K}')$.

However, the MinInc inconsistency value MIV_C presented in [12] does not satisfy the Dominance property. To illustrate this, consider $\tilde{K} = \{a, a \wedge c, \neg a \vee b, a \wedge c \wedge (\neg a \vee b), \neg b\}$, then $\text{MI}(\tilde{K}) = \{\{a, \neg a \vee b, \neg b\}, \{a \wedge c, \neg a \vee b, \neg b\}, \{a \wedge c \wedge (\neg a \vee b), \neg b\}\}$ and $a \wedge c \wedge (\neg a \vee b) \vdash \neg a \vee b$. But $\text{MIV}_C(\tilde{K}, a \wedge c \wedge (\neg a \vee b)) = \frac{1}{2} < \text{MIV}_C(\tilde{K}, \neg a \vee b) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. So, the Dominance property is not considered as a mandatory property to characterize the blame measure.

The other four properties can be derived from (D1)–(D5), as shown by the following lemmas.

LEMMA 3.2

If **Blame** satisfies (D1)–(D5), and **Inc** satisfies (I1)–(I4), then

- (1) **Blame** must satisfy **Distribution**, **Minimality**, and **Decomposability**.
- (2) **Blame** must satisfy **Symmetry** in the case of classical knowledge bases.

Note that in [12], the Distribution and Minimality properties are given in terms of classical knowledge bases while in this article the corresponding properties, Distribution (D4) and Innocence (D2), are presented in terms of minimal inconsistent knowledge bases. There are no significant differences between them in nature if **Inc** satisfies (I1). Because the principle of proportionality proposed in this article focuses on the proportional distribution of the amount of inconsistency of a minimal inconsistent knowledge base among all its formulas, we prefer to present these properties in terms of minimal inconsistent knowledge bases.

On the other hand, Symmetry is a property derived from Shapley inconsistency value [11]. It does not always hold in the case of prioritized knowledge bases intuitively. To illustrate this, consider $K_3 = \langle \{a, \neg b\}, \{\neg a, b\} \rangle$, then for all $K' \subseteq K_3$ s.t. $a, \neg a \notin K'$, $\text{MI}(K' \cup \{a\}) = \text{MI}(K' \cup \{\neg a\})$. However, intuitively, $\text{Blame}(K_3, a) \neq \text{Blame}(K_3, \neg a)$. To replace the Symmetry property, we propose the Fairness property below for the prioritized knowledge bases as follows:

- **Fairness:** $\forall K \in \mathcal{K}_L$, if $\text{Sig}(\text{Opp}(M, \alpha^P)) = \text{Sig}(\text{Opp}(M, \beta^P))$ for each $M \in \text{MI}(K)$, then $\text{Blame}(K, \alpha^P) = \text{Blame}(K, \beta^P)$.

Evidently, the Fairness property can be derived from combination of (D1) and (D5).

PROPOSITION 3.1

If **Blame** satisfies (D1) and (D5), then **Blame** satisfies **Fairness**.

Overall, Proportionality property (D5) essentially distinguishes the characterization of inconsistency measures presented in this article from those presented in [11, 12].

More importantly, if a measure **Blame** satisfies (D1)–(D5), then **Blame** can be formally defined with the following definition.

DEFINITION 3.2 (The Blame of a Formula for Inconsistency)

Let K be a prioritized knowledge base and α^P a formula belonging to K . Then the blame of α^P for the inconsistency of K , denoted $\text{Blame}_P(K, \alpha^P)$, is defined as follows:

$$\text{Blame}_P(K, \alpha^P) = \sum_{M \in \text{MI}(K) \text{ s.t. } \alpha^P \in M} \frac{\text{Sig}(\text{Opp}(M, \alpha^P))}{\sum_{\beta^P \in M} \text{Sig}(\text{Opp}(M, \beta^P))} \times \text{Inc}(M).$$

PROPOSITION 3.2

$\text{Blame}(K, \alpha^P)$ satisfies (D1)–(D5) if and only if

$$\text{Blame}(K, \alpha^P) = \text{Blame}_P(K, \alpha^P).$$

This proposition shows that the measure defined above can be completely characterized by (D1)–(D5). Note that the instances of $\text{Blame}_P(K, \alpha^P)$ depend on the choice of Sig and Inc . For different types of knowledge bases, we may choose appropriate definitions of Sig and Inc to reflect the nature of the knowledge bases (i.e. classical, Type -I and Type-II prioritized etc).

4 Measuring the blame of each formula for the inconsistency of a knowledge base

In this section, we provide a family of measures for the blame of each formula for the inconsistency of classical knowledge bases, Type-I prioritized knowledge bases, and Type-II prioritized knowledge bases, respectively.

4.1 The blame of each formula for the inconsistency of a classical knowledge base

Each formula of a classical knowledge base has the same priority level, so the number of formulas belonging to a classical knowledge base may be considered as the significance of that base.

DEFINITION 4.1 (Significance function Sig_c)

The significance function for classical knowledge bases, denoted Sig_c , is a function $\text{Sig}_c: \tilde{\mathcal{K}}_{\mathcal{L}} \mapsto \mathbf{N}$ such that $\forall \tilde{K} \in \tilde{\mathcal{K}}_{\mathcal{L}}$,

$$\text{Sig}_c(\tilde{K}) = |\tilde{K}|.$$

PROPOSITION 4.1

The significance function Sig_c satisfies the properties (S1)–(S4).

Recall the properties (I1)–(I5) for characterizing the measures for the amount of inconsistency of a knowledge base. In the case of classical knowledge bases, (I3) can be ignored since $|\tilde{M}_1| = |\tilde{M}_2|$ iff $\text{Sig}_c(\tilde{M}_1) = \text{Sig}_c(\tilde{M}_2)$ for any two minimal inconsistent classical knowledge bases \tilde{M}_1 and \tilde{M}_2 .

The MI inconsistency measure I_{MI} defined in [12], for measuring the amount of inconsistency of a classical knowledge base, does not satisfy properties (I4) and (I5), since $\text{I}_{\text{MI}}(\tilde{M}) = 1$ for any minimal inconsistent classical knowledge base \tilde{M} . To address this, we make use of the inconsistency measure Inc_c as the measure for the amount of inconsistency for classical knowledge bases, which is one of the particular weighted MI inconsistency measures for classical knowledge bases presented in our previous article [22].

DEFINITION 4.2 (The inconsistency measure Inc_c [22])

The inconsistency measure for classical knowledge bases, denoted as Inc_c , is a function $\text{Inc}_c: \tilde{\mathcal{K}}_{\mathcal{L}} \mapsto \mathbf{R}$ such that $\forall \tilde{K} \in \tilde{\mathcal{K}}_{\mathcal{L}}$,

$$\text{Inc}_c(\tilde{K}) = \sum_{\tilde{M} \in \text{MI}(\tilde{K})} \text{Inc}_c(\tilde{M}),$$

where $\text{Inc}_c(\tilde{M}) = \frac{1}{|\tilde{M}|}$ for each $\tilde{M} \in \text{MI}(\tilde{K})$.

It has been shown in [22] that Inc_c satisfies the instance of (I2), (I4) and (I5) in the case of classical knowledge bases. However, the following proposition shows Inc_c satisfies all of the five properties for characterizing the measures for the amount of inconsistency of a prioritized knowledge base, if we consider each classical knowledge base as a special kind of prioritized knowledge base.

PROPOSITION 4.2

The inconsistency measure Inc_c satisfies the properties (I1), (I2), (I3), (I4) and (I5).

Based on the significance function Sig_c and the inconsistency measure Inc_c , we can instantiate Blame_p in the case of classical knowledge bases as follows.

DEFINITION 4.3 (The Blame of each formula for the Inconsistency Blame_c)

The blame of each formula for the inconsistency of a classical knowledge base, denoted as Blame_c , is a function $\text{Blame}_c: \tilde{\mathcal{K}}_{\mathcal{L}} \times \mathcal{L} \mapsto \mathbf{R}$ such that $\forall \tilde{K} \in \tilde{\mathcal{K}}_{\mathcal{L}}, \forall \alpha \in \tilde{K}$,

$$\text{Blame}_c(\tilde{K}, \alpha) = \sum_{\tilde{M} \in \text{MI}(\tilde{K})} \text{Blame}_c(\tilde{M}, \alpha),$$

where

$$\text{Blame}_c(\tilde{M}, \alpha) = \frac{\text{Sig}_c(\text{Opp}(\tilde{M}, \alpha))}{\sum_{\beta \in \tilde{M}} \text{Sig}_c(\text{Opp}(\tilde{M}, \beta))} \times \text{Inc}_c(\tilde{M})$$

for each minimal inconsistent subset \tilde{M} of \tilde{K} .

$$\text{Evidently, } \forall \alpha \in \tilde{K}, \text{Sig}_c(\text{Opp}(\tilde{M}, \alpha)) = |\text{Opp}(\tilde{M}, \alpha)| = \begin{cases} |\tilde{M}| - 1, & \text{if } \{\alpha\} \subset \tilde{M} \\ 1, & \text{if } \{\alpha\} = \tilde{M} \\ 0, & \text{if } \alpha \notin \tilde{M} \end{cases}$$

for each $\tilde{M} \in \text{MI}(\tilde{K})$, moreover,

$$\forall \alpha, \beta \in \tilde{M}, |\text{Opp}(\tilde{M}, \alpha)| = |\text{Opp}(\tilde{M}, \beta)| = \begin{cases} |\tilde{M}| - 1, & \text{if } |\tilde{M}| > 1 \\ 1, & \text{if } |\tilde{M}| = 1 \end{cases},$$

then

$$\frac{|\text{Opp}(\tilde{M}, \alpha)|}{\sum_{\beta \in \tilde{M}} |\text{Opp}(\tilde{M}, \beta)|} = \begin{cases} \frac{1}{|\tilde{M}|}, & \text{if } \alpha \in \tilde{M} \\ 0, & \text{if } \alpha \notin \tilde{M} \end{cases}.$$

Therefore,

$$\text{Blame}_c(\tilde{K}, \alpha) = \sum_{\tilde{M} \in \text{MI}(\tilde{K}) \text{ s.t. } \alpha \in \tilde{M}} \frac{1}{|\tilde{M}|^2}.$$

Note that this measure for the blame of each formula guided by (D1)–(D5) is exactly the Type-II weighted MinInc inconsistency value MIV_{wc} presented in [22]. In some sense, this coincidence illustrates that the proportionality is an underlying principle of measures for the blame of each formula for inconsistency in the case of classical knowledge bases.

EXAMPLE 4.1

Consider $\tilde{K}_4 = \{a \wedge \neg a, b, \neg b, \neg b \vee c, \neg c, d\}$. Then the set of minimal inconsistent subsets of \tilde{K}_4 is $\text{MI}(\tilde{K}_4) = \{\tilde{M}_1, \tilde{M}_2, \tilde{M}_3\}$, where

$$\tilde{M}_1 = \{a \wedge \neg a\}, \tilde{M}_2 = \{b, \neg b\}, \tilde{M}_3 = \{b, \neg b \vee c, \neg c\}.$$

So,

$$\begin{aligned}\text{Blame}_c(\tilde{K}_4, a \wedge \neg a) &= 1, \quad \text{Blame}_c(\tilde{K}_4, b) = \frac{13}{36} \\ \text{Blame}_c(\tilde{K}_4, \neg b) &= \frac{1}{4}, \quad \text{Blame}_c(\tilde{K}_4, \neg b \vee c) = \frac{1}{9} \\ \text{Blame}_c(\tilde{K}_4, \neg c) &= \frac{1}{9}, \quad \text{Blame}_c(\tilde{K}_4, d) = 0.\end{aligned}$$

COROLLARY 4.1

Blame_c satisfies the properties (D1)–(D5).

It has been shown in [22] that the Type-II weighted MinInc inconsistency value MIV_{Wc} satisfies the property of **Symmetry**, **Minimality** and **Distribution**, and so does Blame_c . In addition, the following corollary shows that Blame_c satisfies the property of Decomposability. That is, Blame_c satisfies the most expected properties for a measure of the blame of each formula in inconsistency presented in [12].

COROLLARY 4.2

Blame_c satisfies the properties of **Symmetry**, **Minimality**, **Distribution** and **Decomposability**.

These two propositions show that Blame_c is an anticipated measure for the blame of each formula for the inconsistency of a classical knowledge base.

Note that if we need to consider the dominance property, we may adopt the vectorial measure presented in [22] as Inc to define a blame measure in a similar way.

Compared to the MI inconsistency measure I_{MI} , the inconsistency measure Inc_c satisfies (I4) and (I5). This means that Inc_c supports the intuition illustrated by the Lottery Paradox. So, Inc_c is more intuitive than I_{MI} . On the other hand, as illustrated by Corollary 4.2, the blame measure Blame_c satisfies the four properties satisfied by the MinInc inconsistency value MIV_c . But $\sum_{\alpha \in \tilde{M}} \text{MIV}_c(\tilde{M}, \alpha) = \text{I}_{\text{MI}}(\tilde{M}) = 1$, it contradicts the intuition illustrated by the Lottery Paradox. Therefore, Blame_c is more intuitive than MIV_c .

4.2 The blame of each formula for the inconsistency of type-I prioritized knowledge bases

Each individual formula of a Type-I prioritized knowledge base is attached with a numerical valuation in $[0, 1]$, which represents the relative importance or the reliability of the corresponding knowledge. Intuitively, the priority levels of formulas of a Type-I prioritized knowledge base should be taken into account in measuring the significance of that base.

DEFINITION 4.4 (The max-significance function Sig_{max})

The max-significance function for Type-I prioritized knowledge bases, denoted Sig_{max} , is a function $\text{Sig}_{\text{max}}: \hat{\mathcal{K}}_{\mathcal{L}} \mapsto \mathbf{R}$ such that $\forall \hat{K} \in \hat{\mathcal{K}}_{\mathcal{L}}$,

$$\text{Sig}_{\text{max}}(\hat{K}) = \eta_{\text{max}}(\hat{K}) \times \text{Sig}_c(K^*),$$

where $\eta_{\text{max}}(\hat{K}) = \max\{P_I(\phi) \mid (\phi, P_I(\phi)) \in \hat{K}\}$.

DEFINITION 4.5 (The mean-significance function Sig_{mean})

The mean-significance function for Type-I prioritized knowledge bases, denoted Sig_{mean} , is a function $\text{Sig}_{\text{mean}}: \widehat{\mathcal{K}}_{\mathcal{L}} \mapsto \mathbf{R}$ such that $\forall \widehat{K} \in \widehat{\mathcal{K}}_{\mathcal{L}}$,

$$\text{Sig}_{\text{mean}}(\widehat{K}) = \eta_{\text{mean}}(\widehat{K}) \times \text{Sig}_{\mathbf{c}}(K^*),$$

$$\text{where } \eta_{\text{mean}}(\widehat{K}) = \frac{1}{|\widehat{K}|} \sum_{(\phi, P_1(\phi)) \in \widehat{K}} P_1(\phi).$$

The max-significance function insists on taking the priority level of the most preferred formula of a prioritized knowledge base as the significance of that base. In contrast, the mean-significance function emphasizes that the significance of a prioritized knowledge base depends on the average value of the priority levels of the formulas of the base. Note that both Sig_{max} and Sig_{mean} comply with $\text{Sig}_{\mathbf{c}}$ in the case of classical knowledge bases, i.e. if $\widehat{K} = \{(\alpha_1, 1), \dots, (\alpha_n, 1)\}$, then $\text{Sig}_{\text{max}}(\widehat{K}) = \text{Sig}_{\text{mean}}(\widehat{K}) = \text{Sig}_{\mathbf{c}}(K^*) = |\widehat{K}|$.

PROPOSITION 4.3

Both Sig_{max} and Sig_{mean} satisfy the properties (S1)–(S4).

The priority levels of formulas of a Type-I prioritized knowledge base should also be taken into account in measuring the amount of inconsistency of that base. However, most techniques proposed so far for measuring inconsistency focused on certain types of Type-I knowledge bases, such as possibilistic knowledge bases. Roughly speaking, there are two representative kinds of inconsistency measures for prioritized knowledge bases, i.e. model-based measures and syntax-based measures. The model-based measures for a knowledge bases such as significance functions defined for quasi-possibilistic logic [5] were often built upon some (e.g. possibilistic) distribution over models of that base, which is always associated with the priority of levels of formulas in that base. In contrast, the syntax-based measures such as the α -cut defined in possibilistic logic [6] were built upon the formulas of that knowledge base directly. In this article, we focus on syntax-based measures since the proposals based on minimal inconsistent subset are syntax sensitive.

DEFINITION 4.6 (α -cut [6])

Let \widehat{K} be a possibilistic knowledge base and $\alpha \in [0, 1]$. The α -cut of \widehat{K} , denoted $\widehat{K}_{\geq \alpha}$, is defined as

$$\widehat{K}_{\geq \alpha} = \{(\phi, P_1(\phi)) \in \widehat{K} \mid P_1(\phi) \geq \alpha\}.$$

DEFINITION 4.7 (The inconsistency degree [6])

The inconsistency degree of a possibilistic knowledge base \widehat{K} , denoted $\text{Inc}_0(\widehat{K})$, is defined as the maximum value of α such that the α -cut of \widehat{K} is inconsistent, i.e.

$$\text{Inc}_0(\widehat{K}) = \max\{\alpha \mid \widehat{K}_{\geq \alpha} \text{ is inconsistent}\}.$$

Evidently, we can get

$$\text{Inc}_0(\widehat{K}) = \max\{\text{Inc}_0(\widehat{M}) \mid \widehat{M} \in \text{MI}(\widehat{K})\}.$$

EXAMPLE 4.2

Consider $\widehat{K}_5 = \{(a, 0.8), (\neg a \vee b, 0.6), (\neg b, 0.9), (c, 0.5)\}$. Then

$$\begin{aligned} \widehat{K}_{5 \geq 0.9} &= \{(\neg b, 0.9)\}, \\ \widehat{K}_{5 \geq 0.6} &= \{(a, 0.8), (\neg a \vee b, 0.6), (\neg b, 0.9)\}, \\ \widehat{K}_{5 \geq 0.5} &= \widehat{K}_5. \end{aligned}$$

So,

$$\text{Inc}_0(\widehat{K}_5) = 0.6.$$

However, the inconsistency measure Inc_0 does not satisfy the expected properties for a desirable measure of the amount of inconsistency except property (I2). To illustrate this, let us consider an inconsistent prioritized knowledge base corresponding to a classical knowledge base, $\widehat{K} = \{(\alpha_1, 1), \dots, (\alpha_n, 1)\}$ s.t. $|\text{MI}(\widehat{K})| > 1$, then $\text{Inc}_0(\widehat{K}) = 1 < \sum_{\widehat{M} \in \text{MI}(\widehat{K})} \text{Inc}_0(\widehat{M})$. Moreover, $\forall \widehat{M} \in \text{MI}(\widehat{K})$,

$\text{Inc}_0(\widehat{M}) = 1$. It signifies that as the size of the minimal inconsistent subset increases, the amount of inconsistency does not attenuate.

As argued earlier, the significance of a minimal inconsistent subset of a prioritized knowledge base should be considered when measuring the amount of inconsistency of that minimal inconsistent subset. Roughly speaking, (I3) (the property of Monotony w.r.t. Significance) requires that as the significance of a minimal inconsistent knowledge base with a given size increases, the amount of inconsistency cannot decrease. In this sense, we may consider $\text{Inc}(\widehat{M}) = f(\text{Sig}(\widehat{M}))$, where f is a monotonic function. One of such simple functions is the linear function, i.e. $\text{Inc}(\widehat{M}) = \lambda \times \text{Sig}(\widehat{M})$, where λ is a constant related to the size of \widehat{M} . On the other hand, the classical minimal inconsistent knowledge bases of a given size can be considered as the most preferred minimal inconsistent knowledge bases with the size, i.e. the classical minimal inconsistent knowledge bases have the maximal significance among the minimal inconsistent prioritized knowledge base with the same size (as required by (S4)). Then $\text{Inc}(M^*) = \lambda \times |\widehat{M}|$. Therefore, $\lambda = \frac{\text{Inc}(M^*)}{|\widehat{M}|}$. So, we may define an inconsistency measure for minimal inconsistent knowledge bases as follows:

$$\text{Inc}(\widehat{M}) = \frac{\text{Inc}(M^*)}{|\widehat{M}|} \times \text{Sig}(\widehat{M}).$$

Under this guidance, corresponding to the two significance functions defined above, we define two inconsistency measures for the amount of inconsistency of a prioritized knowledge base as follows.

DEFINITION 4.8 (The max-inconsistency measure Inc_{\max})

The max-inconsistency measure for Type-I prioritized knowledge bases, denoted Inc_{\max} , is a function $\text{Inc}_{\max} : \widehat{\mathcal{K}}_{\mathcal{L}} \mapsto \mathbf{R}$ such that $\forall \widehat{K} \in \widehat{\mathcal{K}}_{\mathcal{L}}$,

$$\text{Inc}_{\max}(\widehat{K}) = \sum_{\widehat{M} \in \text{MI}(\widehat{K})} \text{Inc}_{\max}(\widehat{M}),$$

where $\text{Inc}_{\max}(\widehat{M}) = \frac{\text{Sig}_{\max}(\widehat{M})}{|\widehat{M}|} \times \text{Inc}_c(M^*)$ for each $\widehat{M} \in \text{MI}(\widehat{K})$.

Note that $\text{Inc}_{\max}(\widehat{M})$ is proportional to the significance of \widehat{M} . Moreover, $\text{Inc}_{\max}(\widehat{M})$ can be simplified as a product of $\text{Inc}_c(M^*)$ and a factor $\eta_{\max}(\widehat{M})$, i.e. $\text{Inc}_{\max}(\widehat{M}) = \eta_{\max}(\widehat{M}) \times \text{Inc}_c(M^*)$. This also signifies that $\text{Inc}_{\max}(\widehat{M})$ considers the the priority level of the most preferred formulas as well as the size of \widehat{M} .

PROPOSITION 4.4

The inconsistency measure Inc_{\max} satisfies properties (I1)–(I5).

DEFINITION 4.9 (The mean-inconsistency measure Inc_{mean})

The mean-inconsistency measure for Type-I prioritized knowledge bases, denoted as Inc_{mean} , is a function $\text{Inc}_{\text{mean}}: \widehat{\mathcal{K}}_{\mathcal{L}} \mapsto \mathbf{R}$ such that $\forall \widehat{K} \in \widehat{\mathcal{K}}_{\mathcal{L}}$,

$$\text{Inc}_{\text{mean}}(\widehat{K}) = \sum_{\widehat{M} \in \text{MI}(\widehat{K})} \text{Inc}_{\text{mean}}(\widehat{M}),$$

where $\text{Inc}_{\text{mean}}(\widehat{M}) = \frac{\text{Sig}_{\text{mean}}(\widehat{M})}{|\widehat{M}|} \times \text{Inc}_{\text{c}}(M^*)$ for each $\widehat{M} \in \text{MI}(\widehat{K})$.

Similar to Inc_{max} , $\text{Inc}_{\text{mean}}(\widehat{M})$ can be also simplified as a product of $\text{Inc}_{\text{c}}(M^*)$ and a factor $\eta_{\text{mean}}(\widehat{M})$. Compared to Inc_{max} , $\text{Inc}_{\text{mean}}(\widehat{M})$ considers the average of the priority levels of all formulas of \widehat{M} as well as the size of \widehat{M} .

PROPOSITION 4.5

The inconsistency measure Inc_{mean} satisfies properties (I1)–(I5).

EXAMPLE 4.3

Consider $\widehat{K}_6 = \{(a, 0.7), (\neg a, 0.5), (\neg b \vee c, 0.9), (b, 0.5), (\neg c, 0.4), (d, 0.9)\}$. Then $\text{MI}(\widehat{K}_6) = \{\widehat{M}_1, \widehat{M}_2\}$, where

$$\widehat{M}_1 = \{(a, 0.7), (\neg a, 0.5)\}, \quad \widehat{M}_2 = \{(\neg b \vee c, 0.9), (b, 0.5), (\neg c, 0.4)\}.$$

So,

$$\begin{aligned} \text{Inc}_{\text{max}}(\widehat{M}_1) &= 0.35, \quad \text{Inc}_{\text{max}}(\widehat{M}_2) = 0.3, \quad \text{Inc}_{\text{max}}(\widehat{K}_6) = 0.65. \\ \text{Inc}_{\text{mean}}(\widehat{M}_1) &= 0.3, \quad \text{Inc}_{\text{mean}}(\widehat{M}_2) = 0.2, \quad \text{Inc}_{\text{mean}}(\widehat{K}_6) = 0.5. \end{aligned}$$

Based on the two inconsistency measures, we can instantiate **Blame_p** in the case of Type-I prioritized knowledge bases as follows:

DEFINITION 4.10 (The Blame of each formula for the Inconsistency **Blame_{max}**)

Let \widehat{K} be a Type-I prioritized knowledge base. The blame of each formula belonging to \widehat{K} for the inconsistency of \widehat{K} under Inc_{max} , denoted **Blame_{max}**, is a function such that $\forall \alpha^P \in \widehat{K}$,

$$\text{Blame}_{\text{max}}(\widehat{K}, \alpha^P) = \sum_{\widehat{M} \in \text{MI}(\widehat{K})} \text{Blame}_{\text{max}}(\widehat{M}, \alpha^P),$$

where

$$\text{Blame}_{\text{max}}(\widehat{M}, \alpha^P) = \frac{\text{Sig}_{\text{max}}(\text{Opp}(\widehat{M}, \alpha^P))}{\sum_{\beta^P \in \widehat{M}} \text{Sig}_{\text{max}}(\text{Opp}(\widehat{M}, \beta^P))} \times \text{Inc}_{\text{max}}(\widehat{M})$$

for each minimal inconsistent subset \widehat{M} of \widehat{K} .

DEFINITION 4.11 (The Blame of each formula for the Inconsistency **Blame_{mean}**)

Let \widehat{K} be a Type-I prioritized knowledge base. The blame of each formula belonging to \widehat{K} for the inconsistency of \widehat{K} under Inc_{mean} , denoted **Blame_{mean}**, is a function such that $\forall \alpha^P \in \widehat{K}$,

$$\text{Blame}_{\text{mean}}(\widehat{K}, \alpha^P) = \sum_{\widehat{M} \in \text{MI}(\widehat{K})} \text{Blame}_{\text{mean}}(\widehat{M}, \alpha^P),$$

where

$$\text{Blame}_{\text{mean}}(\widehat{M}, \alpha^P) = \frac{\text{Sig}_{\text{mean}}(\text{Opp}(\widehat{M}, \alpha^P))}{\sum_{\beta^P \in \widehat{M}} \text{Sig}_{\text{mean}}(\text{Opp}(\widehat{M}, \beta^P))} \times \text{Inc}_{\text{mean}}(\widehat{M})$$

for each minimal inconsistent subset \widehat{M} of \widehat{K} .

EXAMPLE 4.4

Consider $\widehat{K}_7 = \{(a, 0.6), (\neg a, 0.4), (\neg a \vee c, 0.9), (b, 0.5), (\neg c, 0.3), (d, 0.9)\}$. Then $\text{MI}(\widehat{K}_7) = \{\widehat{M}_1, \widehat{M}_2\}$, where

$$\widehat{M}_1 = \{(a, 0.6), (\neg a, 0.4)\}, \quad \widehat{M}_2 = \{(\neg a \vee c, 0.9), (a, 0.6), (\neg c, 0.3)\}.$$

So,

$$\begin{aligned} \text{Inc}_{\text{max}}(\widehat{M}_1) &= 0.3, & \text{Inc}_{\text{max}}(\widehat{M}_2) &= 0.3, & \text{Inc}_{\text{max}}(\widehat{K}_6) &= 0.6. \\ \text{Inc}_{\text{mean}}(\widehat{M}_1) &= 0.25, & \text{Inc}_{\text{mean}}(\widehat{M}_2) &= 0.2, & \text{Inc}_{\text{mean}}(\widehat{K}_6) &= 0.45. \end{aligned}$$

$$\begin{aligned} \text{Blame}_{\text{max}}(\widehat{K}_7, (a, 0.6)) &= 0.23 & \text{Blame}_{\text{max}}(\widehat{K}_7, (\neg a, 0.4)) &= 0.18 \\ \text{Blame}_{\text{max}}(\widehat{K}_7, (\neg c, 0.3)) &= 0.11 & \text{Blame}_{\text{max}}(\widehat{K}_7, (\neg a \vee c, 0.9)) &= 0.075 \\ \text{Blame}_{\text{max}}(\widehat{K}_7, (b, 0.5)) &= 0 & \text{Blame}_{\text{max}}(\widehat{K}_7, (d, 0.9)) &= 0 \\ \text{Blame}_{\text{mean}}(\widehat{K}_7, (a, 0.6)) &= 0.17 & \text{Blame}_{\text{mean}}(\widehat{K}_7, (\neg a, 0.4)) &= 0.15 \\ \text{Blame}_{\text{mean}}(\widehat{K}_7, (\neg c, 0.3)) &= 0.08 & \text{Blame}_{\text{mean}}(\widehat{K}_7, (\neg a \vee c, 0.9)) &= 0.05 \\ \text{Blame}_{\text{mean}}(\widehat{K}_7, (b, 0.5)) &= 0 & \text{Blame}_{\text{mean}}(\widehat{K}_7, (d, 0.9)) &= 0 \end{aligned}$$

COROLLARY 4.3

Both $\text{Blame}_{\text{max}}$ and $\text{Blame}_{\text{mean}}$ satisfy properties (D1)–(D5).

This proposition shows that both $\text{Blame}_{\text{max}}$ and $\text{Blame}_{\text{mean}}$ are reasonable measures of the blame of each formula for the inconsistency of a Type-I prioritized knowledge base. It accords with the principle of proportionality as well as the viewpoint of minimal inconsistent subset as the purest form of inconsistency.

The following corollary also illustrates that both $\text{Blame}_{\text{max}}$ and $\text{Blame}_{\text{mean}}$ satisfy the most properties presented in [11, 12] as well as the property of Fairness.

COROLLARY 4.4

Both $\text{Blame}_{\text{max}}$ and $\text{Blame}_{\text{mean}}$ satisfy the properties of **Distribution**, **Minimality**, **Decomposability** and **Fairness**.

4.3 The blame of each formula for the inconsistency of type-II prioritized knowledge bases

As introduced earlier, a Type-II prioritized knowledge base K , with n levels of qualitative priorities, is represented as a n -tuple of sets of formulas, i.e.

$$K = \langle K(1), \dots, K(n) \rangle,$$

where each $K(i)$ is a (possibly empty) set of formulas at the i -th qualitative priority level. Generally, it is often hard to use a single value to measure the significance of such knowledge bases, since it

is very subtle to replace each qualitative level with a numerical weight. Moreover, the weights may not be explained intuitively in many cases [21]. For example, consider $\langle \{a\}, \emptyset, \{\neg a\} \rangle$, if we assign weight 9 to a (at a relatively high level) and 1 to $\neg a$ (at a relatively low level), we do not think that the degree of importance of a is exactly 9 times of that of $\neg a$. Intuitively, the significance of a Type-II prioritized knowledge base K depends on the priority levels of its formulas as well as the number of formulas at each priority level. In particular, the priority level of a formula of K is more of a relative comparative value of the significance of this formula w.r.t. that of the other formulas in the same base, therefore, this relative comparative value should be retained when defining the significance of such a base. To address these, in this section, we use a vector to measure the significance of K instead of a single value.

DEFINITION 4.12 (The Significance Vector Sig_v)

Let $K = \langle K(1), \dots, K(n) \rangle$ be a Type-II prioritized knowledge base. The significance vector for K , denoted $\text{Sig}_v(K)$, is defined as

$$\text{Sig}_v(K) = (\text{Sig}_c(K(1)), \dots, \text{Sig}_c(K(n))).$$

Based on the lexicographical ordering relation (denoted as \preceq),¹ the position of each $\text{Sig}_c(K(i))$ in $\text{Sig}_v(K)$ embodies the priority level of formulas in $K(i)$, while $\text{Sig}_c(K(i))$ represents the cardinality of set $K(i)$. This way, we hope that $\text{Sig}_v(K)$ can indeed capture the significance of K . From now on, we call $\text{Sig}_c(K(i))$ the i -th level significance of K , and abbreviate it as $\text{Sig}_v^{(i)}(K)$.

On the other hand, $\text{Sig}_v(K^*) = \left(\sum_{i=1}^n \text{Sig}_c(K(i)), 0, \dots, 0 \right)$, since a classical knowledge base has only one level, then we have $\text{Sig}_v(K) \preceq \text{Sig}_v(K^*)$. However, the following proposition illustrates that Sig_v satisfies the expected properties.

PROPOSITION 4.6

Sig_v satisfies properties (S1)–(S4).

Correspondingly, we also use a vector instead of a single value to measure the inconsistency of a Type-II prioritized knowledge base.

DEFINITION 4.13 (The inconsistency measure Inc_v)

Let $K = \langle K(1), \dots, K(n) \rangle$ be a Type-II prioritized knowledge base. The inconsistency measure for K , denoted as $\text{Inc}_v(K)$, is defined as

$$\text{Inc}_v(K) = \sum_{M \in \text{MI}(K)} \text{Inc}_v(M),$$

where $\text{Inc}_v(M) = \frac{\text{Sig}_v(M)}{|\text{MI}|} \times \text{Inc}_c(M^*)$ for each $M \in \text{MI}(K)$.

Especially, we call $\frac{\text{Sig}_v^{(i)}(M)}{|\text{MI}|} \times \text{Inc}_c(M^*)$ the i -th level inconsistency amount of M , and abbreviate it as $\text{Inc}_v^{(i)}(M)$.

¹A lexicographical ordering relation between any two vectors with the same size is given as follows. Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$ be two vectors. Suppose that $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$. Then the lexicographical ordering relation \preceq defines $\mathbf{u} \preceq \mathbf{v}$ iff

- (1) $\mathbf{u} = \mathbf{v}$, or
- (2) there exists $k \leq n$ s. t. $u_k < v_k$ and $u_i = v_i$ for each $i < k$.

Furthermore, $\mathbf{u} < \mathbf{v}$ iff $\mathbf{u} \preceq \mathbf{v}$ and $\mathbf{u} \neq \mathbf{v}$.

Note that $\text{Inc}_v(\mathbf{M})$ is the product of the significance of \mathbf{M} and a factor $\frac{\text{Inc}_c(\mathbf{M}^*)}{|\mathbf{M}|}$. That is, the amount of inconsistency of \mathbf{M} is proportional to the significance of \mathbf{M} .

PROPOSITION 4.7

Inc_v satisfies properties (I1)–(I5).

EXAMPLE 4.5

Consider $\mathbf{K}_8 = \langle \{a, b, \neg d \vee \neg c\}, \{\neg a, c\}, \{d, \neg b\} \rangle$. Then the minimal inconsistent subsets of \mathbf{K}_8 are

$$\text{MI}(\mathbf{K}_8) = \{\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3\},$$

where

$$\mathbf{M}_1 = \langle \{a\}, \{\neg a\}, \emptyset \rangle, \mathbf{M}_2 = \langle \{b\}, \emptyset, \{\neg b\} \rangle, \mathbf{M}_3 = \langle \{\neg d \vee \neg c\}, \{c\}, \{d\} \rangle.$$

So,

$$\begin{aligned} \text{Inc}_v(\mathbf{M}_1) &= \left(\frac{1}{4}, \frac{1}{4}, 0\right), \quad \text{Inc}_v(\mathbf{M}_2) = \left(\frac{1}{4}, 0, \frac{1}{4}\right), \\ \text{Inc}_v(\mathbf{M}_3) &= \left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right), \quad \text{Inc}_v(\mathbf{K}_8) = \left(\frac{11}{18}, \frac{13}{36}, \frac{13}{36}\right). \end{aligned}$$

To instantiate Blame_P in the case of Type-II prioritized knowledge bases, we need to consider some special features of Type-II prioritized knowledge bases. Note that we use a vector rather than a single value to measure inconsistency for Type-II prioritized knowledge bases. This vectorial measurement considers the qualitative information about priority of each formula involved in inconsistency as well as the number of formulas at each priority level. In this sense, it is more appropriate to capture the inconsistency than a single value. On the other hand, this vectorial measure also divides the whole inconsistency of a prioritized knowledge base into n levels. Correspondingly, we need to focus on the blame of each formula for each level inconsistency.

DEFINITION 4.14 (The blame of each formula for the k -th level inconsistency)

Let $\mathbf{K} = \langle \mathbf{K}(1), \dots, \mathbf{K}(n) \rangle$ be a Type-II prioritized knowledge base. Then for each $1 \leq k \leq n$, the blame of each formula of \mathbf{K} for the k -th level inconsistency of \mathbf{K} , denoted $\text{Blame}_v^{(k)}$, is defined as follows:

$$\forall \alpha \in \mathbf{K}, \text{Blame}_v^{(k)}(\mathbf{K}, \alpha) = \sum_{\mathbf{M} \in \text{MI}(\mathbf{K})} \text{Blame}_v^{(k)}(\mathbf{M}, \alpha),$$

where

$$\text{Blame}_v^{(k)}(\mathbf{M}, \alpha) = \begin{cases} \frac{\text{Sig}_v^{(k)}(\text{Opp}(\mathbf{M}, \alpha))}{\sum_{\beta \in \mathbf{M}} \text{Sig}_v^{(k)}(\text{Opp}(\mathbf{M}, \beta))} \times \text{Inc}_v^{(k)}(\mathbf{M}), & \text{if } |\mathbf{M}(k)| > 0, \\ 0, & \text{if } |\mathbf{M}(k)| = 0. \end{cases}$$

for each $\mathbf{M} \in \text{MI}(\mathbf{K})$.

Essentially, for each $1 \leq k \leq n$, $\text{Blame}_v^{(k)}(\mathbf{M}, \alpha)$ captures the contribution made by α through involving the formulas at the k -level for the inconsistency of \mathbf{M} .

EXAMPLE 4.6

Consider a minimal inconsistent knowledge base $M_9 = \langle \{a\}, \{\neg a \vee b\}, \{\neg b\} \rangle$. Then $\text{Inc}_v(M_9) = \left(\frac{1}{9}, \frac{1}{9}, \frac{1}{9}\right)$. And the opposed formulas to each formula of M_9 are

$$\begin{aligned} \text{Opp}(M_9, a) &= \langle \emptyset, \{\neg a \vee b\}, \{\neg b\} \rangle, & \text{Opp}(M_9, \neg a \vee b) &= \langle \{a\}, \emptyset, \{\neg b\} \rangle, \\ \text{Opp}(M_9, \neg b) &= \langle \{a\}, \{\neg a \vee b\}, \emptyset \rangle. \end{aligned}$$

So,

$$\text{Blame}_v^{(1)}(M_9, a) = 0, \text{Blame}_v^{(2)}(M_9, a) = \frac{1}{18}, \text{Blame}_v^{(3)}(M_9, a) = \frac{1}{18}.$$

Note that $\text{Blame}_v^{(1)}(M_9, a) = 0$. It implies that formula a is not responsible for the first level inconsistency, since formula a only involves one formula with the second level ($\neg a \vee b$) and another formula with the third level ($\neg b$) in inconsistency. This is consistent with our intuition.

DEFINITION 4.15 (The Blame of each Formula for Inconsistency)

Let $K = \langle K(1), \dots, K(n) \rangle$ be a Type-II prioritized knowledge base. The blame of each formula of K for the inconsistency of K , denoted Blame_v , is defined as follows:

$$\forall \alpha \in K, \text{Blame}_v(K, \alpha) = (\text{Blame}_v^{(1)}(K, \alpha), \dots, \text{Blame}_v^{(n)}(K, \alpha)),$$

where $\text{Blame}_v^{(k)}(K, \alpha)$ is the blame of α to the k -th level inconsistency of K for each $1 \leq k \leq n$.

Evidently, $\text{Blame}_v(K, \alpha)$ captures globally the contribution made by α to the inconsistency of K .

EXAMPLE 4.7

Consider $K_{10} = \langle \{a, b, \neg d \vee \neg a\}, \{\neg a, c\}, \{d, \neg b\} \rangle$. Then the set of minimal inconsistent subsets of K_{10} is

$$\text{MI}(K_{10}) = \{M_1, M_2, M_3\},$$

where

$$M_1 = \langle \{a\}, \{\neg a\}, \emptyset \rangle, \quad M_2 = \langle \{b\}, \emptyset, \{\neg b\} \rangle, \quad M_3 = \langle \{a, \neg d \vee \neg a\}, \emptyset, \{d\} \rangle.$$

The blame of each formula for the inconsistency of K_{10} is given as follows:

$$\begin{aligned} \text{Blame}_v(K_{10}, a) &= \left(\frac{1}{18}, \frac{1}{4}, \frac{1}{18}\right), & \text{Blame}_v(K_{10}, \neg a) &= \left(\frac{1}{4}, 0, 0\right), \\ \text{Blame}_v(K_{10}, b) &= \left(0, 0, \frac{1}{4}\right), & \text{Blame}_v(K_{10}, \neg b) &= \left(\frac{1}{4}, 0, 0\right), \\ \text{Blame}_v(K_{10}, \neg d \vee \neg a) &= \left(\frac{1}{18}, 0, \frac{1}{18}\right), & \text{Blame}_v(K_{10}, d) &= \left(\frac{1}{9}, 0, 0\right), \\ \text{Blame}_v(K_{10}, c) &= (0, 0, 0). \end{aligned}$$

The following proposition and the corresponding corollary show that Blame_v is a desirable measure for the blame of each formula of a Type-II prioritized knowledge base for the inconsistency of that base.

PROPOSITION 4.8

Blame_v satisfies (D1)–(D5).

Note that if $M(k) = \emptyset$, then $\text{Blame}_v^{(k)}(M, \alpha) = \text{Sig}_v^{(k)}(\text{Opp}(M, \alpha)) = 0$.

COROLLARY 4.5

Blame_v satisfies the properties of **Distribution**, **Minimality**, **Decomposability** and **Fairness**.

To compare the formulas of a given prioritized knowledge base in terms of their blames for the inconsistency of that base, we define a relation over the prioritized knowledge base as follows.

DEFINITION 4.16 (The relation of less blameful than, \leq_B)

Let K be a Type-II prioritized knowledge base. A binary relation on K , denoted as \leq_B , is defined as follows: $\alpha \leq_B \beta$ if and only if

$$\text{Blame}_v(K, \alpha) \leq \text{Blame}_v(K, \beta).$$

Further, $\alpha <_B \beta$ if $\alpha \leq_B \beta$ and $\beta \not\leq_B \alpha$. $\alpha \simeq_B \beta$ if $\alpha \leq_B \beta$ and $\beta \leq_B \alpha$. We say that α is less blameful for the inconsistency in K than β if $\alpha <_B \beta$.

Note that the relation \leq_B is a total ordering on K .

EXAMPLE 4.8

Consider K_{10} again. Then

$$c <_B b <_B \neg d \vee \neg a <_B a <_B d <_B \neg a \simeq_B \neg b.$$

According to this ordering relation, b is less blameful for the inconsistency in K than $\neg b$. Generally, we may change $\neg b$ rather than b to resolve the inconsistency $b \wedge \neg b$ based on this relation.

Note that a prioritized knowledge base mentioned in this article is associated with a prioritization function, i.e. each formula is attached with a numerical or qualitative priority level. However, in some applications, the prioritization over a knowledge base is given by a binary preference relation between formulas in the base. Moreover, a relational preference is considered as one of the two main families of mathematical models describing the preference over a set of candidates [18].

A stratified knowledge base can be considered as a representative of such kind of prioritized knowledge base. Roughly speaking, a stratified knowledge base is a classical knowledge base \tilde{K} coupled with a total pre-order relation \ll on \tilde{K} . From the given pre-order relation \ll , its corresponding classical knowledge base can be stratified as $\tilde{K} = (S_1, \dots, S_n)$, where S_i contains all the minimal formulas of set $\bigcup_{j=i}^n S_j$ w.r.t. \ll . Each S_i is called a stratum of K and is non-empty.

The measure **Blame_v** for the blame of each formula for the inconsistency of a Type-II prioritized knowledge base can be extended to the stratified knowledge bases as discussed above, because **Blame_v** only concerns with the relative preference among formulas in the context of a given knowledge base. In a given stratified knowledge base $K = (S_1, \dots, S_n)$, we may consider S_i as a set of formulas at the i -th qualitative priority level in the context of K , i.e. we may use $\langle S_1, \dots, S_n \rangle$ instead of (S_1, \dots, S_n) .

5 An application in requirements engineering

Here we use an example on analyzing the requirements for updating an existing software system to illustrate the application of the approach to measuring the blames of formulas in inconsistency presented in this article.

EXAMPLE 5.1

Consider a scenario for eliciting requirements for updating an existing software system.

- (a) Stakeholder A: the delegate of the seller of the new system, provides the following requirements:
 - (a1) The system-to-be should be open, that is, the system-to-be could be extended easily;
 - (a2) The user interface of the system-to-be should be fashionably designed;
 - (a3) The system-to-be should be developed based on the newest development techniques.
- (b) Stakeholder B: the delegate of the users of the existing system, provides the following requirements:
 - (b1) The system-to-be should be developed based on the techniques used in the existing system;
 - (b2) The user interface of the system-to-be should maintain the style of the existing system;
 - (b3) The system-to-be should be secure.
- (c) Constraint:
 - (c1) To guarantee the security of the system-to-be, its openness should not be considered.

After balancing the requirements of Stakeholder A against that of Stakeholder B, the requirements analysts assign the priority levels *High* to (a3) and (b3), *Medium* to (a2) and (b1) and *Low* to (a1) and (b2), respectively. On the other hand, the requirements analysts assign the level of *High* to the constraint.

Suppose that we

- use the predicate $\text{Fash}(\text{int_f})$ to denote that the interface is fashionable;
- use the predicate $\text{Open}(\text{sys})$ to denote that the system is open;
- use the predicate $\text{New}(\text{sys})$ to denote that the system will be developed based on the newest techniques;
- use the predicate $\text{Secu}(\text{sys})$ to denote that the system is secure.

Then we use a Type-II knowledge base

$$\begin{aligned} K_R = \{ & \{\text{New}(\text{sys}), \text{Secu}(\text{sys}), \text{Secu}(\text{sys}) \rightarrow \neg \text{Open}(\text{sys})\}, \\ & \{\text{Fash}(\text{int_f}), \neg \text{New}(\text{sys})\}, \{\text{Open}(\text{sys}), \neg \text{Fash}(\text{int_f})\} \} \end{aligned}$$

to represent the requirements above. Evidently, we draw the following inconsistencies from these requirements:

$$\begin{aligned} K_R \vdash & \text{New}(\text{sys}) \wedge \neg \text{New}(\text{sys}), \\ K_R \vdash & \text{Fash}(\text{int_f}) \wedge \neg \text{Fash}(\text{int_f}), \\ K_R \vdash & \text{Open}(\text{sys}) \wedge \neg \text{Open}(\text{sys}). \end{aligned}$$

To resolve the inconsistencies in K_R , some requirements need to be abandoned or to be changed. However, neither Stakeholder A nor Stakeholder B is willing to make some concessions. Therefore, it becomes necessary first to identify the blame of each requirement involving in the inconsistencies of K_R .

The set of all the minimal inconsistent subsets of K_R is $\text{MI}(K_R) = \{M_1, M_2, M_3\}$, where

$$\begin{aligned} M_1 &= \{\{\text{New}(\text{sys})\}, \{\neg \text{New}(\text{sys})\}, \emptyset\}, \\ M_2 &= \{\{\text{Secu}(\text{sys}), \text{Secu}(\text{sys}) \rightarrow \neg \text{Open}(\text{sys})\}, \emptyset, \{\text{Open}(\text{sys})\}\}, \\ M_3 &= \{\emptyset, \{\text{Fash}(\text{int_f})\}, \{\neg \text{Fash}(\text{int_f})\}\}. \end{aligned}$$

Then the blame of each requirement for the inconsistency of K_R is calculated as follows:

$$\begin{aligned} \text{Blame}_v(K_R, \text{New}(\text{sys})) &= \left(0, \frac{1}{4}, 0\right) & \text{Blame}_v(K_R, \neg \text{New}(\text{sys})) &= \left(\frac{1}{4}, 0, 0\right) \\ \text{Blame}_v(K_R, \text{Secu}(\text{sys})) &= \left(\frac{1}{18}, 0, \frac{1}{18}\right) & \text{Blame}_v(K_R, \text{Open}(\text{sys})) &= \left(\frac{1}{9}, 0, 0\right) \\ \text{Blame}_v(K_R, \text{Fash}(\text{int_f})) &= \left(0, 0, \frac{1}{4}\right) & \text{Blame}_v(K_R, \neg \text{Fash}(\text{int_f})) &= \left(0, \frac{1}{4}, 0\right) \end{aligned}$$

$$\text{Blame}_v(K_R, \text{Secu}(\text{sys}) \rightarrow \neg \text{Open}(\text{sys})) = \left(\frac{1}{18}, 0, \frac{1}{18}\right)$$

Following this, we obtain the following ordering:

$$\begin{aligned} \text{Fash}(\text{int_f}) &<_B \neg \text{Fash}(\text{int_f}) \simeq_B \text{New}(\text{sys}) <_B \text{Secu}(\text{sys}) \simeq_B \\ \text{Secu}(\text{sys}) \rightarrow \neg \text{Open}(\text{sys}) &<_B \text{Open}(\text{sys}) <_B \neg \text{New}(\text{sys}) \end{aligned}$$

In particular, for each inconsistency, we have

$$\begin{aligned} \text{Secu}(\text{sys}) &\simeq_B \text{Secu}(\text{sys}) \rightarrow \neg \text{Open}(\text{sys}) <_B \text{Open}(\text{sys}) \\ \text{Fash}(\text{int_f}) &<_B \neg \text{Fash}(\text{int_f}) \\ \text{New}(\text{sys}) &<_B \neg \text{New}(\text{sys}). \end{aligned}$$

Based on this result, requirements analysts may persuade Stakeholder B to change (b1) and (b2). At the same time, as a concession, Stakeholder A should change (a1) if Stakeholder B is willing to go with requirements analysts's suggestion.

In contrast, assume that the developers do not take the priority of each requirement into account in identifying the blame of each requirement for the inconsistency of the requirements set, instead they use a classical knowledge base

$$\begin{aligned} \tilde{K}_R &= \{\text{Open}(\text{sys}), \text{Fash}(\text{int_f}), \text{New}(\text{sys}), \neg \text{New}(\text{sys}), \\ &\quad \neg \text{Fash}(\text{int_f}), \text{Secu}(\text{sys}), \text{Secu}(\text{sys}) \rightarrow \neg \text{Open}(\text{sys})\} \end{aligned}$$

to represent the requirements set above. Then it is very difficult to make a distinction between the blames of two formulas contradicting each other by using the MinInc inconsistency values presented in [12], since for each minimal inconsistent subset, its formulas have the same value of blame, i.e.

$$\begin{aligned} \text{MIV}_C(\tilde{K}_R, \text{New}(\text{sys})) &= \frac{1}{2}, & \text{MIV}_C(\tilde{K}_R, \neg \text{New}(\text{sys})) &= \frac{1}{2}, \\ \text{MIV}_C(\tilde{K}_R, \text{Secu}(\text{sys})) &= \frac{1}{3}, & \text{MIV}_C(\tilde{K}_R, \text{Open}(\text{sys})) &= \frac{1}{3}, \\ \text{MIV}_C(\tilde{K}_R, \text{Secu}(\text{sys}) \rightarrow \neg \text{Open}(\text{sys})) &= \frac{1}{3}, & & \\ \text{MIV}_C(\tilde{K}_R, \text{Fash}(\text{int_f})) &= \frac{1}{2}, & \text{MIV}_C(\tilde{K}_R, \neg \text{Fash}(\text{int_f})) &= \frac{1}{2}. \end{aligned}$$

To conclude, the measures we proposed in this article are better suited to identify blames of formulas involved in inconsistency than existing measures.

6 Related work

Measuring inconsistency in knowledge bases has received considerable attention in computer science as well as in artificial intelligence recently. Many approaches have been proposed accordingly. In this article, we concentrated on prioritized knowledge bases and presented a family of measures for the blames of formulas in inconsistency of a prioritized knowledge base by using the minimal inconsistent subsets of the base. In the following, we compare our measures with some of closely related approaches.

Most of the approaches proposed so far are mainly concerned with measuring inconsistencies in knowledge bases [10]. There are relative few techniques for identifying the degree of blame or responsibility of each formula for the inconsistency of a knowledge base. Hunter and Konieczny presented two approaches to measuring the degree of blame of individual formulas in inconsistency of a classical knowledge base in [11] and [12], respectively. The first approach focuses on the distribution of the measures of inconsistency for a whole knowledge base among formulas by using a cooperative game theory model, i.e. Shapley inconsistency value [11]. However, we have argued that Shapley value is inappropriate for modeling the blame of each formula for the inconsistency of a prioritized knowledge base. Moreover, the symmetry property, one of the four properties completely characterizing the Shapley value, does not hold in the case of prioritized knowledge bases.

The second approach presented in [12, 13] emphasizes the importance of defining the measure of the blames of formulas of a knowledge base from the minimal inconsistent subsets of that base directly. The MinInc inconsistency values [12] and the scoring function [13] can be considered as the most representative work of the second approach. However, as pointed out in [12], the scoring function is very sketchy for measuring the blame of each formula, since it does not consider the size of each minimal inconsistent subset.

The MinInc inconsistency value presented in [12] can be considered as one of the most representative proposals of the second approach. Our measures presented in this article comply with the second approach in upholding the viewpoint of minimal inconsistent subsets of a knowledge base as the purest form of inconsistency in that base. In this sense, these approaches presented in this article may be considered as an extension of the second approach.

However, the following aspects distinguish our measures from the MinInc inconsistency value. First of all, the MinInc inconsistency value focuses on classical knowledge bases. In contrast, the family of measures presented in this article focuses on prioritized knowledge bases as well as classical knowledge bases. Second, the MinInc inconsistency value MIV_C insists that the amount of inconsistency in a minimal inconsistent knowledge base is shared equally among all the formulas of that base. It distributes $\frac{1}{|M|}$ to each formula of a minimal inconsistent subset M . In contrast, we combined the principle of proportionality and the view of minimal inconsistent subsets of a knowledge base as the purest form of inconsistency of that base [25]. The principle of proportionality is central to characterize the blame of each formula for the inconsistency of a prioritized knowledge base. Roughly, the blame of each individual formula for the inconsistency is determined by the opposed formulas to the formula in causing the inconsistency. Third, we present some more intuitive properties to characterize the measure of inconsistency for a prioritized knowledge bases in terms of minimal inconsistent subset. Moreover, in the case of classical knowledge bases, a combination of some (not all) new properties can derive all the properties presented in [12] to capture a basic inconsistency measure except the property of Dominance. However, as illustrated earlier, the property of Dominance does not hold for the MI inconsistency measure I_{MI} , which was considered as a basic inconsistency measure in [12]. Also, it is inappropriate to describe the measures based on minimal

inconsistent subsets if we consider the amount of inconsistency of a knowledge base as the sum of the amounts of inconsistency of minimal inconsistent subsets.

On the other hand, in the case of classical knowledge bases, we use the inconsistency measure Inc_C presented in [22] to derive the measures for the blame of each formula guided by the five properties (D1)–(D5). The measure Inc_C states that $\text{Inc}_C(\tilde{M}) = \frac{1}{|\tilde{M}|}$ for each $\tilde{M} \in \text{MI}(\tilde{K})$. Actually, it complies with the maximal η -consistency presented by Knight [15] in upholding the intuition that bigger size means a smaller amount of the inconsistency. According to the inconsistency measure of maximal η -consistency, a minimal inconsistent classical knowledge base of n ($n > 1$) size is maximally $\frac{n-1}{n}$ -consistent.

Note that the inconsistency measures for prioritized knowledge bases presented in this article take the priority level of formulas into account. In some sense, these measures consider the significance or strength of inconsistency as well as the degree of inconsistency. In particular, in the case of Type-I prioritized knowledge bases, Inc_{\max} and Inc_{mean} agree with the significance functions for quasi-possibilistic logic presented in [5] in taking the strength of conflicts on literals into account. However, the significance functions defined in [5] are model-based measures, i.e. these were built upon a mass assignment over the set of (quasi-possibilistic) models. In contrast, Inc_{\max} and Inc_{mean} were defined from the priority level of formulas as well as the minimal inconsistent subsets directly, that is, there are syntax-based measures. Moreover, these measures are characterized by properties in terms of minimal inconsistent subsets.

7 Conclusions

We have presented a family of measures for the blame of each formula in a prioritized knowledge base in terms of minimal inconsistent subsets of that prioritized knowledge base. This article presented the following contributions to measuring inconsistency for a knowledge base:

- We presented a set of properties to characterize a desirable measure for the amount of inconsistency of a prioritized knowledge base defined from the minimal inconsistent subsets of that base, including MinInc Additivity, Contradiction, Monotony with significance, Attenuation with the size and Almost consistency.
- Motivated by the principle of proportionality, we presented a set of properties to characterize a desirable measure for the blame of each formula in inconsistency of a minimal inconsistent prioritized knowledge base, including Accumulation, Innocence, Necessity, MinInc Distribution and Proportionality.
- We presented a family of measures for the blame of each formula in inconsistency of a prioritized knowledge base under guidance of the principle of proportionality.
- We showed that the measures defined in this article satisfy all the expected properties.
- We used a simple but explanatory example in requirements engineering to illustrate the potential application of the measures for the blames of formulas in inconsistency presented in this article.

As pointed out earlier, the Shapley inconsistency value [11] is inappropriate to model the blame of formulas in inconsistency of a strictly prioritized knowledge base. How to characterize the blame of each formula for the inconsistency of a prioritized knowledge base by using some other appropriate social models will be the main direction for our future work.

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Appendix

PROOF OF LEMMA 3.1

PROOF. Let \tilde{K} and \tilde{K}' be two classical knowledge bases.

- Consistency: Suppose that \tilde{K} is consistent, then $\text{MI}(\tilde{K}) = \emptyset$. By (I1), $\text{Inc}(\tilde{K}) = \sum_{\tilde{M} \in \text{MI}(\tilde{K})} \text{Inc}(\tilde{M}) = 0$.
- Monotony: Obviously, $\text{MI}(\tilde{K}) \subseteq \text{MI}(\tilde{K} \cup \tilde{K}')$, so $\text{Inc}(\tilde{K}) \leq \text{Inc}(\tilde{K} \cup \tilde{K}')$ according to (I1) and (I2).
- Free Formula Independence: If α is a free formula of $\tilde{K} \cup \{\alpha\}$, then $\text{MI}(\tilde{K} \cup \{\alpha\}) = \text{MI}(\tilde{K})$. By (I1), $\text{Inc}(\tilde{K} \cup \{\alpha\}) = \text{Inc}(\tilde{K})$. ■

PROOF OF LEMMA 3.2

PROOF. (1) Let K and K' be two knowledge bases.

- Distribution: By (D1),

$$\forall \alpha^P \in K, \text{Blame}(K, \alpha^P) = \sum_{M \in \text{MI}(K)} \text{Blame}(M, \alpha^P).$$

Then

$$\begin{aligned} \sum_{\alpha^P \in K} \text{Blame}(K, \alpha) &= \sum_{\alpha^P \in K} \sum_{M \in \text{MI}(K)} \text{Blame}(M, \alpha^P) \\ &= \sum_{M \in \text{MI}(K)} \sum_{\alpha^P \in K} \text{Blame}(M, \alpha). \end{aligned}$$

By (D4), $\sum_{\alpha^P \in M} \text{Blame}(M, \alpha) = \text{Inc}(M)$.By (I1), $\sum_{M \in \text{MI}(K)} \text{Inc}(M) = \text{Inc}(K)$. So,

$$\sum_{\alpha^P \in K} \text{Blame}(K, \alpha^P) = \text{Inc}(K).$$

- Minimality: If α^P is a free formula of K , then from (D1) and (D2),

$$\text{Blame}(K, \alpha^P) = \sum_{M \in \text{MI}(K)} \text{Blame}(M, \alpha^P) = 0.$$

- Decomposability: From (D1),

$$\text{Blame}(K \cup K', \alpha^P) = \sum_{M \in \text{MI}(K \cup K')} \text{Blame}(M, \alpha).$$

Suppose that $\text{MI}(K \cup K') = \text{MI}(K) \oplus \text{MI}(K')$, then

$$\begin{aligned} &\sum_{M \in \text{MI}(K \cup K')} \text{Blame}(M, \alpha^P) \\ &= \sum_{M \in \text{MI}(K)} \text{Blame}(M, \alpha^P) + \sum_{M' \in \text{MI}(K')} \text{Blame}(M', \alpha^P) \\ &= \text{Blame}(K, \alpha^P) + \text{Blame}(K', \alpha^P). \end{aligned}$$

(2) Let \tilde{K} be classical knowledge base.

- Symmetry: For such α and β , $\forall \tilde{M} \in \text{MI}(\tilde{K})$ and $\alpha \in \tilde{M}$,

- if $\beta \in \tilde{M}$, let $\tilde{K}' = \text{Opp}(\tilde{M}, \alpha) - \{\beta\}$, then

$$\text{Opp}(\tilde{M}, \beta) = \tilde{K}' \cup \{\alpha\}, \text{Opp}(\tilde{M}, \alpha) = \tilde{K}' \cup \{\beta\}.$$

So,

$$\text{Sig}(\text{Opp}(\tilde{M}, \beta)) = \text{Sig}(\text{Opp}(\tilde{M}, \alpha)).$$

According to (D5),

$$\text{Blame}(\tilde{M}, \alpha) = \text{Blame}(\tilde{M}, \beta).$$

- if $\beta \notin \tilde{M}$, let $\tilde{M}' = \tilde{M} \cup \{\beta\} - \{\alpha\}$, then $\tilde{M}' \in \text{MI}(\tilde{K})$. So,

$$\text{Opp}(\tilde{M}, \alpha) = \text{Opp}(\tilde{M}', \beta) \text{ and } |\tilde{M}| = |\tilde{M}'|.$$

Therefore,

$$\text{Blame}(\tilde{M}, \alpha) = \text{Blame}(\tilde{M}', \beta).$$

Otherwise, if $\tilde{M}' \not\models \perp$, then $\text{Inc}(\tilde{M}) > 0 = \text{Inc}(\tilde{M}')$. (By (I2) and (I1)). If $\exists \tilde{M}'' \subset \tilde{M}$ such that $\tilde{M}'' \in \text{MI}(\tilde{M}')$, then

$$\text{Inc}(\tilde{M}) < \text{Inc}(\tilde{M}'') \leq \text{Inc}(\tilde{M}')$$

since $|\tilde{M}| > |\tilde{M}''|$ (By (I4) and (I1)). Both contradict $\text{Inc}(\tilde{M}) = \text{Inc}(\tilde{M}')$.

Further, by (D1), $\text{Blame}(\tilde{K}, \alpha) = \text{Blame}(\tilde{K}, \beta)$. ■

PROOF OF PROPOSITION 3.1

PROOF. Let K be knowledge base. Suppose that $\forall M \in \text{MI}(K)$, $\text{Sig}(\text{Opp}(M, \alpha^P)) = \text{Sig}(\text{Opp}(M, \beta^P))$. Then from (D5),

$$\text{Blame}(M, \alpha^P) = c(M) \times \text{Sig}(\text{Opp}(M, \alpha^P)) = \text{Blame}(M, \beta^P).$$

Further,

$$\sum_{M \in \text{MI}(K)} \text{Blame}(M, \alpha^P) = \sum_{M \in \text{MI}(K)} \text{Blame}(M, \beta^P).$$

By (D1), we can get

$$\text{Blame}(K, \alpha^P) = \text{Blame}(K, \beta^P). \quad \blacksquare$$

PROOF OF PROPOSITION 3.2

PROOF. ' \Leftarrow '. $\text{Blame}_P(K, \alpha^P)$ satisfies (D1–D5):

(D1) It follows the definition of Blame_P .

(D2) $\forall M \in \mathcal{M}_L$, $\alpha^P \notin M$, then $\text{Opp}(M, \alpha^P) = \emptyset$. By (S1), $\text{Sig}(\text{Opp}(M, \alpha^P)) = 0$. So, $\text{Blame}_P(M, \alpha^P) = 0$.

(D3) $\forall M \in \mathcal{M}_L$, $\alpha^P \in M$, then $\text{Opp}(M, \alpha^P) \neq \emptyset$. By (S1), $\text{Sig}(\text{Opp}(M, \alpha^P)) > 0$. So, $\text{Blame}_P(M, \alpha^P) > 0$.

(D4) $\forall M \in \mathcal{M}_L$, $\sum_{\alpha^P \in M} \text{Blame}_P(M, \alpha^P) = \sum_{\alpha^P \in M} \frac{\text{Sig}(\text{Opp}(M, \alpha^P))}{\sum_{\beta^P \in M} \text{Sig}(\text{Opp}(M, \beta^P))} \times \text{Inc}(M) = \text{Inc}(M)$.

(D5) Given $M \in \mathcal{M}_{\mathcal{L}}$, let $c(M) = \frac{\text{Inc}(M)}{\sum_{\beta^P \in M} \text{Sig}(\text{Opp}(M, \beta^P))}$, then $\forall \alpha^P \in M$, $\text{Blame}_P(M, \alpha^P) = c(M) \times$

$$\text{Sig}(\text{Opp}(M, \alpha^P)).$$

' \implies '. If $\text{MI}(K) = \emptyset$, from (D1), $\text{Blame}(K, \alpha^P) = 0 = \text{Blame}_P(K, \alpha^P)$.

If $\text{MI}(K) \neq \emptyset$, from (D5) and (D3), we can get $\forall M \in \text{MI}(K)$,

$$\sum_{\beta^P \in M} \text{Blame}(M, \beta^P) = c(M) \times \sum_{\beta^P \in M} \text{Sig}(\text{Opp}(M, \beta^P)) > 0.$$

From (D4), we can get $\forall M \in \text{MI}(K)$,

$$\sum_{\beta^P \in M} \text{Blame}(M, \beta^P) = \text{Inc}(M).$$

Then

$$c(M) = \frac{\text{Inc}(M)}{\sum_{\beta^P \in M} \text{Sig}(\text{Opp}(M, \beta^P))}$$

From (D1) and ((D2)), $\forall \alpha^P \in K$,

$$\text{Blame}(K, \alpha^P) = \sum_{M \in \text{MI}(K)} \text{Blame}(M, \alpha^P) = \sum_{M \in \text{MI}(K) \text{ s.t. } \alpha^P \in M} \text{Blame}(M, \alpha^P),$$

i.e.

$$\text{Blame}(K, \alpha^P) = \sum_{M \in \text{MI}(K) \text{ s.t. } \alpha^P \in M} \frac{\text{Sig}(\text{Opp}(M, \alpha^P))}{\sum_{\beta^P \in M} \text{Sig}(\text{Opp}(M, \beta^P))} \times \text{Inc}(M).$$

■

PROOF OF PROPOSITION 4.1

PROOF. Let \tilde{K} and \tilde{K}' be two classical knowledge bases.

- (S1) $\text{Sig}_c(\tilde{K}) = |\tilde{K}| = 0$ if $\tilde{K} = \emptyset$.
- (S2) $\forall \alpha, \beta \notin \tilde{K}$, $\text{Sig}_c(\tilde{K} \cup \{\alpha\}) = |\tilde{K}| + 1 = \text{Sig}_c(\tilde{K} \cup \{\beta\})$.
- (S3) $\text{Sig}_c(\tilde{K} \cup \tilde{K}') = |\tilde{K} \cup \tilde{K}'| \geq |\tilde{K}| = \text{Sig}_c(\tilde{K})$.
- (S4) $\text{Sig}_c(\tilde{K}) = \text{Sig}_c(K^*)$ because of $\tilde{K} = K^*$.

■

PROOF OF PROPOSITION 4.2

PROOF. Let \tilde{K} be a classical knowledge base. Suppose that \tilde{M}_1 and \tilde{M}_2 are two minimal inconsistent knowledge bases such that $|\tilde{M}_1| < |\tilde{M}_2|$.

- (I1) It follows the definition directly.
- (I2) $\forall \tilde{M}$, $\text{Inc}_c(\tilde{M}) = \frac{1}{|\tilde{M}|} > 0$.
- (I3) It is trivial, since $|\tilde{M}_1| = |\tilde{M}_2|$ iff $\text{Sig}_c(\tilde{M}_1) = \text{Sig}_c(\tilde{M}_2)$ for any two minimal inconsistent classical knowledge bases \tilde{M}_1 and \tilde{M}_2 .
- (I4) $\text{Inc}_c(\tilde{M}_1) = \frac{1}{|\tilde{M}_1|} > \frac{1}{|\tilde{M}_2|} = \text{Inc}_c(\tilde{M}_2)$ since $|\tilde{M}_1| < |\tilde{M}_2|$.
- (I5) $\frac{1}{|\tilde{M}|} \rightarrow 0$ if $|\tilde{M}| \rightarrow +\infty$.

■

PROOF OF COROLLARY 4.1

PROOF. According to Proposition 3.2, Blame_c satisfies the properties (D1)–(D5), since Blame_c is an instantiated measure of Blame_p . In particular, regarding (D5), given $\tilde{M} \in \tilde{\mathcal{M}}_{\mathcal{L}}$, consider $c(\tilde{M}) = \begin{cases} \frac{\text{Inc}_c(\tilde{M})}{|\tilde{M}|(|\tilde{M}|-1)}, & \text{if } |\tilde{M}| > 1 \\ 1, & \text{if } |\tilde{M}| = 1 \end{cases}$, then $\forall \alpha \in \tilde{M}$, $\text{Blame}_c(\tilde{M}, \alpha) = c(\tilde{M}) \times |\text{Opp}(\tilde{M}, \alpha)|$. ■

PROOF OF COROLLARY 4.2

PROOF. It is a direct consequence of Corollary 4.1, Proposition 4.2 and Lemma 3.2. ■

PROOF OF PROPOSITION 4.3

PROOF. Let \hat{K} and \hat{K}' be two knowledge bases.

- (S1) $\text{Sig}_{\max}(\hat{K}) = \text{Sig}_{\text{mean}}(\hat{K}) = 0$ if $\hat{K} = \emptyset$, since $\text{Sig}_c(K^*) = 0$.
 (S2) If $(\alpha, P_I(\alpha)), (\beta, P_I(\beta)) \notin \hat{K}$ such that $P_I(\alpha) \geq P_I(\beta)$, then
- $\eta_{\max}(\hat{K} \cup \{(\alpha, P_I(\alpha))\}) \geq \eta_{\max}(\hat{K} \cup \{(\beta, P_I(\beta))\})$,
 - $\eta_{\text{mean}}(\hat{K} \cup \{(\alpha, P_I(\alpha))\}) \geq \eta_{\text{mean}}(\hat{K} \cup \{(\beta, P_I(\beta))\})$.
- On the other hand, $\text{Sig}_c(K^* \cup \{\alpha\}) = \text{Sig}_c(K^* \cup \{\beta\}) = |\hat{K}| + 1$. So,
- $\text{Sig}_{\max}(\hat{K} \cup \{(\alpha, P_I(\alpha))\}) \geq \text{Sig}_{\max}(\hat{K} \cup \{(\beta, P_I(\beta))\})$,
 - $\text{Sig}_{\text{mean}}(\hat{K} \cup \{(\alpha, P_I(\alpha))\}) \geq \text{Sig}_{\text{mean}}(\hat{K} \cup \{(\beta, P_I(\beta))\})$.
- (S3) Evidently, $\eta_{\max}(\hat{K} \cup \hat{K}') \geq \eta_{\max}(\hat{K})$, and $\text{Sig}_c(K^* \cup K'^*) = |\hat{K} \cup \hat{K}'| \geq |\hat{K}| = \text{Sig}_c(K^*)$, so,

$$\text{Sig}_{\max}(\hat{K} \cup \hat{K}') \geq \text{Sig}_{\max}(\hat{K}).$$

With regard to Sig_{mean} ,

$$\eta_{\text{mean}}(\hat{K} \cup \hat{K}') = \frac{\sum_{(\phi, P_I(\phi)) \in \hat{K}} P_I(\phi) + \sum_{(\psi, P_I(\psi)) \in \hat{K}' - \hat{K}} P_I(\psi)}{|\hat{K} \cup \hat{K}'|},$$

then

$$\begin{aligned} \text{Sig}_{\text{mean}}(\hat{K} \cup \hat{K}') &= \sum_{(\phi, P_I(\phi)) \in \hat{K}} P_I(\phi) + \sum_{(\psi, P_I(\psi)) \in \hat{K}' - \hat{K}} P_I(\psi) \\ &\geq \sum_{(\phi, P_I(\phi)) \in \hat{K}} P_I(\phi) = \text{Sig}_{\text{mean}}(\hat{K}). \end{aligned}$$

- (S4) $\eta_{\text{mean}}(\hat{K}) \leq \eta_{\max}(\hat{K}) \leq 1$, so, $\text{Sig}_{\text{mean}}(\hat{K}) \leq \text{Sig}_{\max}(\hat{K}) \leq \text{Sig}_c(K^*)$. ■

PROOF OF PROPOSITION 4.4

PROOF. Let \hat{K} be a Type-I knowledge base.

- (I1) It follows the definition of Inc_{\max} directly.
 (I2) $\forall M \in \tilde{\mathcal{M}}_{\mathcal{L}}$, $\text{Sig}_{\max}(\hat{M}) > 0$. By Proposition 4.2, $\text{Inc}_c(M^*) > 0$. So, $\text{Inc}_{\max}(\hat{M}) = \frac{\text{Sig}_{\max}(\hat{M})}{|\hat{M}|} \times \text{Inc}_c(M^*) > 0$.
 (I3) Suppose that \hat{M}_1 and \hat{M}_2 are two minimal inconsistent knowledge bases such that $|\hat{M}_1| = |\hat{M}_2|$, then $\text{Inc}_c(M_1^*) = \text{Inc}_c(M_2^*)$ according to Proposition 4.2. Further, if $\text{Sig}_{\max}(\hat{M}_1) \leq \text{Sig}_{\max}(\hat{M}_2)$, then $\text{Inc}_{\max}(\hat{M}_1) \leq \text{Inc}_{\max}(\hat{M}_2)$.

- (I4) Suppose that \widehat{M}_1 and \widehat{M}_2 are two minimal inconsistent knowledge bases such that $\forall(\alpha, P_I(\alpha)), (\beta, P_I(\beta)) \in \widehat{M}_1 \cup \widehat{M}_2, P_I(\alpha) = P_I(\beta)$, then $\eta_{\max}(\widehat{M}_1) = \eta_{\max}(\widehat{M}_2)$. if $|\widehat{M}_1| < |\widehat{M}_2|$, then $\text{Inc}_c(M_1^*) > \text{Inc}_c(M_2^*)$ according to Proposition 4.2. So, $\text{Inc}_{\max}(\widehat{M}_1) > \text{Inc}_{\max}(\widehat{M}_2)$.
- (I5) $\forall \widehat{M} \in \widehat{\mathcal{M}}_{\mathcal{L}}, 0 < \eta_{\max}(\widehat{M}) \leq 1$. By proposition 4.2, $\text{Inc}_c(M^*) \rightarrow 0$ if $|\widehat{M}| \rightarrow 0$. So, $\text{Inc}_{\max}(\widehat{M}) \rightarrow 0$ if $|\widehat{M}| \rightarrow 0$. ■

PROOF OF PROPOSITION 4.5

The proof of this proposition is very similar to that of Proposition 4.4.

PROOF. Let \widehat{K} be a Type-I knowledge base.

- (I1) It follows the definition of Inc_{mean} directly.
- (I2) $\forall M \in \widehat{\mathcal{M}}_{\mathcal{L}}, \text{Sig}_{\text{mean}}(\widehat{M}) > 0$. By Proposition 4.2, $\text{Inc}_c(M^*) > 0$. So, $\text{Inc}_{\text{mean}}(\widehat{M}) = \frac{\text{Sig}_{\text{mean}}(\widehat{M})}{|\widehat{M}|} \times \text{Inc}_c(M^*) > 0$.
- (I3) Suppose that \widehat{M}_1 and \widehat{M}_2 are two minimal inconsistent knowledge bases such that $|\widehat{M}_1| = |\widehat{M}_2|$, then $\text{Inc}_c(M_1^*) = \text{Inc}_c(M_2^*)$ according to Proposition 4.2. Further, if $\text{Sig}_{\text{mean}}(\widehat{M}_1) \leq \text{Sig}_{\text{mean}}(\widehat{M}_2)$, then $\text{Inc}_{\text{mean}}(\widehat{M}_1) \leq \text{Inc}_{\text{mean}}(\widehat{M}_2)$.
- (I4) Suppose that \widehat{M}_1 and \widehat{M}_2 are two minimal inconsistent knowledge bases such that $\forall(\alpha, P_I(\alpha)), (\beta, P_I(\beta)) \in \widehat{M}_1 \cup \widehat{M}_2, P_I(\alpha) = P_I(\beta)$, then $\eta_{\text{mean}}(\widehat{M}_1) = \eta_{\text{mean}}(\widehat{M}_2)$. if $|\widehat{M}_1| < |\widehat{M}_2|$, then $\text{Inc}_c(M_1^*) > \text{Inc}_c(M_2^*)$ according to Proposition 4.2. So, $\text{Inc}_{\text{mean}}(\widehat{M}_1) > \text{Inc}_{\text{mean}}(\widehat{M}_2)$.
- (I5) $\forall \widehat{M} \in \widehat{\mathcal{M}}_{\mathcal{L}}, 0 < \eta_{\text{mean}}(\widehat{M}) \leq 1$. By proposition 4.2, $\text{Inc}_c(M^*) \rightarrow 0$ if $|\widehat{M}| \rightarrow 0$. So, $\text{Inc}_{\text{mean}}(\widehat{M}) \rightarrow 0$ if $|\widehat{M}| \rightarrow 0$. ■

PROOF OF COROLLARY 4.3

PROOF. Both Blame_{\max} and $\text{Blame}_{\text{mean}}$ are instances of Blame_P , from Proposition 3.2, the two measures satisfy properties (D1)–(D5). ■

PROOF OF COROLLARY 4.4

PROOF. It is a direct consequence of Corollary 4.3, Proposition 3.1, Lemma 3.2, Proposition 4.4 and Proposition 4.5. ■

PROOF OF PROPOSITION 4.6

PROOF. Let $K = (K(1), K(2), \dots, K(n))$ be a Type-II prioritized knowledge base.

- (S1) Zero Significance: $\text{Sig}_v(K) = \mathbf{0}$, where $\mathbf{0}$ is the zero vector. \Leftrightarrow for each i , $\text{Sig}_c(K(i)) = 0$. $\Leftrightarrow K = \emptyset$.
- (S2) Preference: Suppose that $\beta^P, \alpha^P \notin K$ and $P_{II}(\alpha) = k \leq P_{II}(\beta) = m$. Then

$$\text{Sig}_v^{(i)}(K \cup \{\beta^P\}) = \begin{cases} \text{Sig}_v^{(i)}(K), & \text{if } i \neq m, \\ \text{Sig}_v^{(i)}(K) + 1, & \text{if } i = m. \end{cases}$$

$$\text{Sig}_v^{(i)}(K \cup \{\alpha^P\}) = \begin{cases} \text{Sig}_v^{(i)}(K), & \text{if } i \neq k, \\ \text{Sig}_v^{(i)}(K) + 1, & \text{if } i = k. \end{cases}$$

So, for each $i < k$, $\text{Sig}_v^{(i)}(K \cup \{\beta^P\}) = \text{Sig}_v^{(i)}(K \cup \{\alpha^P\})$ and $\text{Sig}_v^{(k)}(K \cup \{\beta^P\}) \leq \text{Sig}_v^{(k)}(K \cup \{\alpha^P\})$, i.e.

$$\text{Sig}_v(K \cup \{\beta^P\}) \leq \text{Sig}_v(K \cup \{\alpha^P\}).$$

- (S3) Monotony: $\forall K, K' \in \mathcal{K}_{\mathcal{L}}$, for each $1 \leq i \leq n$, $K(i) \subseteq K(i) \cup K'(i)$, then $\text{Sig}_{\mathbf{v}}^{(i)}(K) \leq \text{Sig}_{\mathbf{v}}^{(i)}(K \cup K')$. Therefore, $\text{Sig}_{\mathbf{v}}(K) \leq \text{Sig}_{\mathbf{v}}(K \cup K')$.
- (S4) Upper Bound: $\forall K \in \mathcal{K}_{\mathcal{L}}$, $\text{Sig}_{\mathbf{v}}^{(1)}(K) \leq \sum_{i=1}^n \text{Sig}_{\mathbf{v}}^{(i)}(K) = \text{Sig}_{\mathbf{v}}^{(1)}(K^*)$. So, $\text{Sig}_{\mathbf{v}}(K) \leq \text{Sig}_{\mathbf{v}}(K^*)$. ■

PROOF OF PROPOSITION 4.7

PROOF. (I1) $\forall K \in \mathcal{K}_{\mathcal{L}}$, $\text{Inc}_{\mathbf{v}}(K) = \sum_{M \in \text{MI}(K)} \text{Inc}_{\mathbf{v}}(M)$. It follows the definition of $\text{Inc}_{\mathbf{v}}$ directly.

- (I2) $\forall M \in \mathcal{M}_{\mathcal{L}}$, $\text{Inc}_{\mathbf{c}}(M^*) > 0$ by Proposition 4.2. On the other hand, $M \neq \emptyset$, then $\mathbf{0} < \text{Sig}_{\mathbf{v}}(M)$ by Proposition 4.6. So, $\mathbf{0} < \text{Inc}_{\mathbf{v}}(M)$.
- (I3) $\forall M_1, M_2 \in \mathcal{M}_{\mathcal{L}}$, suppose that $|M_1| = |M_2|$, then $\text{Inc}_{\mathbf{c}}(M_1^*) = \text{Inc}_{\mathbf{c}}(M_2^*)$. Further, if $\text{Sig}_{\mathbf{v}}(M_1) \leq \text{Sig}_{\mathbf{v}}(M_2)$, then $\text{Inc}_{\mathbf{v}}(M_1) \leq \text{Inc}_{\mathbf{v}}(M_2)$.
- (I4) $\forall M_1, M_2 \in \mathcal{M}_{\mathcal{L}}$ s.t. $\forall \alpha, \beta \in M_1 \cup M_2$, $P_{\Pi}(\alpha) = P_{\Pi}(\beta) = k$, then

$$\frac{\text{Sig}_{\mathbf{v}}^{(i)}(M_1)}{|M_1|} = \frac{\text{Sig}_{\mathbf{v}}^{(i)}(M_2)}{|M_2|} = \begin{cases} 0, & \text{if } i \neq k, \\ 1, & \text{if } i = k. \end{cases}$$

Further, if $|M_1| < |M_2|$, then $\text{Inc}_{\mathbf{c}}(M_1^*) > \text{Inc}_{\mathbf{c}}(M_2^*)$ by Proposition 4.2. So, $\text{Inc}_{\mathbf{v}}(M_2) < \text{Inc}_{\mathbf{v}}(M_1)$.

- (I5) $\forall M \in \mathcal{M}_{\mathcal{L}}$, $0 \leq \frac{\text{Sig}_{\mathbf{v}}^{(i)}(M)}{|M|} \leq 1$ for each $1 \leq i \leq n$. By Proposition 4.2, $\text{Inc}_{\mathbf{c}}(M^*) \rightarrow 0$ if $|M| \rightarrow +\infty$. So, $\lim_{|M| \rightarrow +\infty} \text{Inc}_{\mathbf{v}}(M) = \mathbf{0}$. ■

PROOF OF PROPOSITION 4.8

PROOF. (D1) Accumulation: $\forall K \in \mathcal{K}_{\mathcal{L}}$, $\forall \alpha \in K$, from Definition 4.14, for each $1 \leq k \leq n$, $\text{Blame}_{\mathbf{v}}^{(k)}(K, \alpha) = \sum_{M \in \text{MI}(K)} \text{Blame}_{\mathbf{v}}^{(k)}(M, \alpha)$. So,

$$\text{Blame}_{\mathbf{v}}(K, \alpha) = \sum_{M \in \text{MI}(K)} \text{Blame}_{\mathbf{v}}(M, \alpha).$$

- (D2) Innocence: $\forall M \in \mathcal{M}_{\mathcal{L}}$, if $\alpha \notin M$, then $\text{Opp}(M, \alpha) = \emptyset$. So, $\text{Sig}_{\mathbf{v}}(\text{Opp}(M, \alpha)) = \mathbf{0}$. Therefore, $\text{Blame}_{\mathbf{v}}^{(k)}(M, \alpha) = 0$ for each $1 \leq k \leq n$, i.e.

$$\text{Blame}_{\mathbf{v}}(M, \alpha) = \mathbf{0}.$$

- (D3) Necessity: $\forall M \in \mathcal{M}_{\mathcal{L}}$, $\forall \alpha \in M$, $\text{Opp}(M, \alpha) \neq \emptyset$. Then must exist $1 \leq k \leq n$ such that $\text{Inc}_{\mathbf{v}}^{(k)}(M) > 0$ and $\text{Sig}_{\mathbf{v}}^{(k)}(\text{Opp}(M, \alpha)) > 0$. So, $0 < \text{Blame}_{\mathbf{v}}^{(k)}(M, \alpha)$, i.e.

$$\mathbf{0} < \text{Blame}_{\mathbf{v}}(M, \alpha).$$

- (D4) MinInc Distribution: $\forall M \in \mathcal{M}_{\mathcal{L}}$,

$$\begin{aligned} & \sum_{\alpha \in M} \text{Blame}_{\mathbf{v}}(M, \alpha) \\ &= \left(\sum_{\alpha \in M} \text{Blame}_{\mathbf{v}}^{(1)}(M, \alpha), \sum_{\alpha \in M} \text{Blame}_{\mathbf{v}}^{(2)}(M, \alpha), \dots, \sum_{\alpha \in M} \text{Blame}_{\mathbf{v}}^{(n)}(M, \alpha) \right). \end{aligned}$$

- If $M(i) = \emptyset$, then $\sum_{\alpha \in M} \text{Blame}_V^{(i)}(M, \alpha) = 0 = \text{Inc}_V^{(i)}(M)$.
- If $M(i) \neq \emptyset$, then $\sum_{\alpha \in M} \text{Blame}_V^{(i)}(M, \alpha) = \frac{\sum_{\alpha \in M} \text{Sig}_V^{(i)}(\text{Opp}(M, \alpha))}{\sum_{\beta \in M} \text{Sig}_V^{(i)}(\text{Opp}(M, \beta))} \times \text{Inc}_V^{(i)}(M) = \text{Inc}_V^{(i)}(M)$.

So,

$$\sum_{\alpha \in M} \text{Blame}_V(M, \alpha) = \text{Inc}_V(M).$$

(D5) Proportionality: Given $M = \langle M(1), \dots, M(n) \rangle \in \mathcal{M}_{\mathcal{L}}$, $\forall \alpha \in M$, for each k ($1 \leq k \leq n$), $\text{Blame}_V^{(k)}(M, \alpha) = c_k(M) \times \text{Sig}_V^{(k)}(\text{Opp}(M, \alpha))$, where

$$c_k(M) = \begin{cases} \frac{\text{Inc}_V^{(k)}(M)}{\sum_{\beta \in M} \text{Sig}_V^{(k)}(\text{Opp}(M, \beta))}, & \text{if } M(k) \neq \emptyset \\ 1, & \text{if } M(k) = \emptyset \end{cases},$$

i.e. $\text{Blame}_V(M, \alpha) = \text{Sig}_V(\text{Opp}(M, \alpha)) \cdot C_{n \times n}$, where

$$C_{n \times n} = \begin{bmatrix} c_1(M) & 0 & \dots & 0 \\ 0 & c_2(M) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & c_n(M) \end{bmatrix}.$$

PROOF OF COROLLARY 4.5

PROOF. It is a direct consequence of Proposition 4.8, Lemma 3.2, Proposition 3.1 and Proposition 4.7. ■