

A framework for managing uncertain inputs: an axiomization of rewarding*

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Abstract

The success postulate in belief revision ensures that new evidence (input) is always trusted. However, admitting uncertain input has been questioned by many researchers. Darwiche and Pearl argued that strengths of evidence should be introduced to determine the outcome of belief change, and provided a preliminary definition towards this thought. In this paper, we start with Darwiche and Pearl's idea aiming to develop a framework that can capture the influence of the strengths of inputs with some rational assumptions. To achieve this, we first define epistemic states to represent beliefs attached with strength, and then present a set of postulates to describe the change process on epistemic states that is determined by the strengths of input and establish representation theorems to characterize these postulates. As a result, we obtain a unique rewarding operator which is proved to be a merging operator that is in line with many other works. We also investigate existing postulates on belief merging and compare them with our postulates. In addition, we show that from an epistemic state, a corresponding ordinal conditional function by Spohn can be derived and the result of combining two epistemic states is thus reduced to the result of combining two corresponding ordinal conditional functions proposed by Laverny and Lang. Furthermore, when reduced to the belief revision situation, we prove that our results induce all the Darwiche and Pearl's postulates as well as the Recalcitrance postulate and the Independence postulate.

Keywords: Epistemic state; Rewarding operator; Belief revision; Iterated revision; Ordinal conditional function; Belief Merging

1 Introduction

Belief revision depicts the process that an agent revises its beliefs when new evidence (we call it *input* in the rest of the paper) is received. Often, new input is to some extent

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conflicting with an agent's current beliefs. Therefore, belief revision is a framework to characterize the process of belief change in order to revise the agent's current beliefs to accommodate new evidence and to reach a new consistent set of beliefs. Belief revision can be seen as a kind of prioritized belief merging [9] in the sense that the latest input has the highest priority and so takes the advantage, whilst belief merging formally studies how to aggregate them into a coherent one when multiple belief bases are given. Namely, the basic distinction between revision and merging is that revision is asymmetric w.r.t pieces of information and merging is symmetric.

Most studies on belief revision are based on the AGM postulates [1]. In [19], the AGM postulates are formulated in the propositional setting, denoted as R1-R6. This set of postulates characterizes what a revision operator shall comply. The R1 postulate, also called *success postulate*, requires that the revision result of a belief set K by a proposition μ (new information) should always maintain μ being believed. This postulate has been questioned (e.g., [7, 14, 6], etc), because it is often undesirable in situations where an agent's observation is imprecise or uncertain.

Practically¹, at any point in our life, we have a set of beliefs, and we observe. Generally we trust our observations - but not always. This makes sense: mostly, we trust the evidence of our senses, but sometimes we imagine things, or perform measurements or tests carelessly. Therefore, given that a belief set K and new input μ are not consistent, we cannot completely believe μ despite of its conflict to K , but we need to cooperate them. Of course, in this situation, extra information describing to what extent μ would be trusted, i.e., strength of μ , should be provided, which in fact makes the observation an uncertain input.

This viewpoint was formalized in [11], in which it says, for uncertain inputs, the main question is how to interpret them. Typically there are two interpretations making sense.

Enforced Uncertainty The input is taken as a constraint that must be satisfied by the revision result. This is the common view of revision.

Unreliable input The input is interpreted as an extra piece of information that may be useful or may be not when refining the current state. Hence it is not necessarily kept after revision. This is the focus of the paper.

This issue was also implicitly mentioned by Darwiche and Pearl [8]. In the Future Work Section [8], Darwiche and Pearl argued that revision should allow *the outcome of belief change depends on the strength of evidence triggering the change*. Hence the notions of *evidence strength* and *degree of acceptance* was introduced. In particular, it was stated that a proposition μ is accepted by an epistemic state² Φ to degree m if it takes a piece of evidence $\neg\mu$ with strength m to retract μ from Φ . Formally, they gave the following definition.

Definition 1 [8] *Proposition μ is accepted by an epistemic state Φ to degree m (written $\Phi \models_m \mu$) precisely when*

¹This paragraph is based on a review comment on our Flairs 2009 submission.

²[8] did not clearly define what an epistemic state is, but it can be considered as an agent's current beliefs together with the relative plausibility orderings of worlds which are inconsistent with the current beliefs.

1. $\Phi \not\models \neg\mu$;
2. $\Phi \circ_m \neg\mu \not\models \neg\mu$; and
3. $\Phi \circ_m \neg\mu \not\models \mu$.

Here \circ_m is a revision operator incorporating value m .

However, how to design such operators and how to manage a sequence of revision operators $\circ_{m_1}, \circ_{m_2}, \dots, \circ_{m_n}$ remain to be investigated.

Dropping the success postulate led to non-prioritized revision, *in which no absolute priority is assigned to the new input* [14]. However, in the current non-prioritized revisions, strengths of new inputs do not play important roles, hence Darwiche and Pearl's suggestions are not realized.

In this paper, we follow Darwiche and Pearl's idea to investigate how we can allow the strength of new input to trigger the outcome of belief change. Our starting assumptions are:

- Introduce strengths on prior beliefs as well as on new inputs;
- Remove the success postulate;
- Adopt Darwiche and Pearl's Definition 1 to allow the strengths of beliefs and input to determine the outcome.

From this starting point, we further develop some additional postulates to emphasize the crucial and unique role of strengths in the strength-determination framework. These postulates include: inputs with a zero strength having no effect on prior beliefs (B0), inputs with the same strength having the same influence (B4), repeated inputs reinforcing each other (B5), orders of inputs having no effect on the outcome (B6), etc.

Our initial intention is to allow strengths of prior beliefs and new input to determine the outcome in belief revision, hoping to obtain some new revision operators. However, with a set of rational postulates proposed for this purpose, we instead obtained a unique merging operator characterized by this set of postulates. This operator is in fact a *rewarding* operator in the sense that it does not give a penalty to dissatisfying worlds, but rewards to satisfying worlds. The rewarding property makes this operator differs from most of the existing works, although after normalization, this operator is reduced to a well-known form, ordinal conditional functions combination in [23, 30], etc. Therefore, the most significant contribution of the paper is the characterization of this rewarding operator, which provides an answer to Darwiche and Pearl's problem.

Since our intuition starts from belief revision, but our result is a belief merging operator, to properly reflect this nature, hereafter, we call the framework in this paper a belief change framework.

It should be noted that uncertain inputs can also be represented as a kind of epistemic states, such as, in [25, 2], although generally epistemic states are used to model agent's beliefs. Hence in this paper, we also use epistemic states to represent uncertain inputs. The use of epistemic states to represent new evidence instead of simply using a propositional formula in belief revision has already been adopted by many revision

frameworks where the advantages of this transition are clearly stated in [8, 4, 5, 26, 18, 29, 15], etc.

In this paper, we first give a formal definition of epistemic state and then present a set of postulates, denoted as B0-B6, to characterize operators on iterated belief change. In this framework, both the prior beliefs and new inputs are syntactically represented by epistemic states. We also provide representation theorems for our postulates.

An interesting phenomena in the research on epistemic revision is that almost all the papers on this topic deploy Spohn's ordinal conditional function (OCF) [30] or its variants as illustrative examples for representing prior beliefs and new inputs. In this paper, we examine the ordinal conditional function and its combination rule [23] in our belief change framework. We prove that from an epistemic state, a corresponding ordinal conditional function can be derived and that the result of combining two epistemic states is equivalent to the result of combining two corresponding OCFs. This finding is significant since it provides a justification for the combination rule of ordinal conditional functions proposed in [23], which is the most notable combination rule for OCFs so far.

Furthermore, when reduced to the belief revision situation where new inputs must be accepted in the revised belief set, we prove that our framework can induce Darwiche and Pearl's (DP's) belief revision postulates [8] as well as the Recalcitrance postulate [29], and the Independence postulate [18].

Since our belief change process is characterized by a merging operator, we also investigate the current belief merging operators and their postulates, provide a detailed comparison between our postulates and the existing ones.

The rest of the paper is organized as follows. In Section 2, after briefly introducing preliminaries, we discuss our motivation on belief change through an example. In Section 3, we provide formal definitions for epistemic states and study iterated belief change by uncertain inputs as well as a justification for the combination rule proposed for OCFs. In Section 4, we prove that our postulates can induce all the DP's iterated belief revision postulates. Finally, we discuss related work and conclude the paper in Section 5 and Section 6.

2 Motivation

Preliminaries: We consider a propositional language \mathcal{L} defined on a finite set \mathcal{A} of propositional atoms, which are denoted by p, q, r etc. A proposition ϕ is constructed by propositional atoms with logic connections $\neg, \wedge, \vee, \rightarrow$ in the standard way. An interpretation ω (or possible world) is a function that maps \mathcal{A} onto the set $\{0, 1\}$. The set of all possible interpretations on \mathcal{A} is denoted as W . Function ω can be extended to any proposition in \mathcal{L} in the usual way, $\omega : \mathcal{L} \rightarrow \{0, 1\}$. An interpretation ω is a model of (or satisfies) ϕ iff $\omega(\phi) = 1$, denoted as $\omega \models \phi$. We use $Mod(\phi)$ to denote the set of models for ϕ .

A pre-order \leq defined on any set A is a reflexive and transitive relation over $A \times A$. \leq is total iff for all elements $a, b \in A$, either $a \leq b$ or $b \leq a$ holds. Conventionally, a strict order $<$ and an indifferent relation $=$ can be induced by \leq such that $\forall a, b \in A$, $a < b$ iff $a \leq b$ but $b \not\leq a$, and $a = b$ iff $a \leq b$ and $b \leq a$.

Motivations: We use an example to further demonstrate our motivation.

Example 1 (Derived from Example 17 in [8]) We face a murder trial with two main suspects, **John** and **Mary**. Initially, it appears that the murder was committed by **one person**, hence, our belief can be characterized as: $(John \wedge \neg Mary) \vee (\neg John \wedge Mary)$. As the trial unfolds, however, we receive a very reliable testimony incriminating John, followed by another reliable testimony incriminating Mary. At this point, how can we judge these two pieces of evidence in relation to the one-person theory? If we do not strongly believe in the one-person theory, we should believe that both suspects took part in the murder; whilst if we believe in the one-person theory more strongly than the testimonies, then based on belief revision, we are forced to believe that Mary is the murderer no matter how compelling the evidence against John is (because the evidence incriminating Mary comes later). This is counterintuitive in two accounts: first, whether we should dismiss the testimony against John should depend on how strongly we believe in it compared with how strongly we believe in the one-person theory; second, whether we should dismiss the testimony against John should also depend on how strongly we believe in it compared with how strongly we believe the testimony against Mary (if one-person theory is to be held).

This example is very interesting. First, it shows that without providing the strength of evidence, any belief revision postulates could lead us to the wrong way, i.e. John can be either a murderer or innocent, any postulate favoring the prior belief (i.e. the one-person theory) may let a potential murderer escape (*John*), and any postulate favoring the testimony may convict a potentially innocent person (*John*). Therefore, evidence should also be represented by epistemic states (attached with strengths). The inspiration of using epistemic state to represent evidence could date back to Jeffrey's work on probability conditionalization [17], where Jeffrey wrote:

in belief updating, the representation of the belief state, the representation of the new information and the result of the update should be of the same type.

Following this statement, Jeffrey represented all these three items as probability measures (which is a kind of epistemic state). This idea also inspired Nayak in [28] to represent pieces of evidence as epistemic entrenchment relations (which can be seen as another kind of epistemic state) in belief revision.

Second, the above example shows that the underlying principle of belief revision, i.e. that the most recent evidence has the highest priority, has a major drawback: even if the testimony against *John* is more compelling than that to *Mary*, *Mary* is still the murderer if the one-person theory is believed. Therefore, who is the murderer somehow depends on which evidence arrives last.

The assumption of giving priority to the most recent evidence is also questioned in [9]. To get around this assumption, iterated belief revision is taken as a prioritized merging where a set of evidence is prioritized according to their reliability (strength) rather than the *order* that these pieces of evidence are received. The revised (or the merged) result is a consistent belief set such that when a more reliable piece of evidence is inconsistent with a less reliable piece of evidence, the more reliable evidence should

be preserved in the revised belief set. Similar works on reliability handling in belief merging are also reported in [3, 32], etc.

Now we examine Example 1 again by incorporating the strengths of evidence. Suppose that the prior knowledge, *one-person theory*, and two new inputs, *John is the murderer*, and *Mary is the murderer* (no matter in which order the evidence (inputs) is collected), are available and have strengths α , β and γ respectively. With a rational belief change process, we should find the real murderer(s) according to those strengths. For example, if one-person theory is to be kept (i.e., $\alpha > \max(\beta, \gamma)$), then the murderer is *John* if $\beta > \gamma$; the murderer is *Mary* if $\beta < \gamma$; and we do not know who the murderer is if $\beta = \gamma$. This solution is obviously more intuitive than the result obtained from iterated belief revision where *Mary* has to be the murderer under one-person theory regardless how strong the evidence against *John* is.

From the analysis above, we get the reinforced view that

1. Epistemic states should be used to represent both the prior beliefs and new inputs to manage belief change.
2. It should be the strengths of new inputs, not the order that the inputs are collected, determine the outcome of belief change.

3 Belief Change with Uncertain Inputs

Our approach on belief change with uncertain inputs is inspired by Darwiche and Pearl's idea in Definition 1. That is we want to describe the belief change process that can truly reflect the strengths of prior beliefs and new evidence. Below we first define epistemic states and then give the corresponding postulates to characterize the process.

3.1 Epistemic states

Ordinal conditional function [30] is commonly regarded as a form of epistemic state.

An **ordinal conditional function**, also known as a **ranking function** or a **kappa function**, commonly denoted as κ , is a function from a set of possible worlds to the set of ordinal numbers with its **belief set** defined as $Bel(\kappa) = \varphi$ where $Mod(\varphi) = \{w | \kappa(w) = 0\}$. Value $\kappa(w)$ is understood as the degree of **disbelief** of world w . So the smaller the $\kappa(w)$ value, the more plausible the world is. The ranking value of a proposition μ is defined as:

$$\kappa(\mu) = \min_{w \models \mu} \kappa(w).$$

The combination of two ordinal conditional functions κ_1 and κ_2 is defined in [23] as

$$(\kappa_1 \oplus \kappa_2)(w) = \kappa_1(w) + \kappa_2(w) - \min_{w \in W} (\kappa_1(w) + \kappa_2(w)) \quad (1)$$

This is applicable only when $\min_{w \in W} (\kappa_1(w) + \kappa_2(w)) < +\infty$.

We can see that there is a normalization step in the combination of OCFs to make the minimal worlds have kappa value 0. However, when modeling the belief change

process, we want to solely concentrate on the nature of the *changing* process, and ignore the normalization. So we do not simply use OCFs as our epistemic states but define epistemic states as follows.

Definition 2 An **epistemic state** Φ is a mapping from W to $Z \cup \{-\infty, \infty\}$ where Z is the set of integers.

Obviously, this definition follows the spirit of OCF and the epistemic state defined in [26] (in which it was defined as a mapping from W to the set of ordinals, and such a definition was also implied in [38]).

Based on Definition 2, if a possible world w is assigned with a larger integer than another world w' , then it is interpreted as that w is believed more plausible than w' .

Nevertheless, two *syntactically* different epistemic states could have the same *semantic* meanings, as stated below.

Definition 3 Two epistemic states Φ and Ψ are considered *semantically equivalent* if and only if $\exists k \in Z$, s.t., $\forall w \in W$, $\Phi(w) = \Psi(w) + k$.

That is, the value assigned to w in an epistemic state only has a relative meaning w.r.t. values assigned to other possible worlds in the same state. Hence after normalization, we can get a semantically equivalent epistemic state. For the sake of clear presentation of postulates, unless otherwise specified, we do not consider normalizing semantically equivalent epistemic states, and just treat them as a single epistemic state.

Definition 4 Let Φ be an epistemic state, the **belief set** of Φ , denoted as $Bel(\Phi)$, is defined as $Bel(\Phi) = \psi$ where $Mod(\psi) = \min(W, \leq_\Phi)$. Here \leq_Φ is a total pre-order relation on W such that $w_1 \leq_\Phi w_2$ iff $\Phi(w_1) \geq \Phi(w_2)$.

Here we can see that the belief set derived from an OCF or an epistemic state defined in [26] is the same as that in Definition 4, i.e., the belief set has all the most plausible worlds as its models, although in [26], the most plausible worlds are assigned value 0 while in this paper, the most plausible worlds are assigned with the largest values comparing to the other worlds. In addition, we have that ignoring the normalization step will not affect the belief sets of the resulted epistemic state based on Equation 1.

Another major difference between our epistemic state and some other definitions [26] is that the range of our epistemic states is $Z \cup \{-\infty, \infty\}$ instead of ordinals. Once again, these two differences enable us to avoid the normalization step.

Φ can be extended to proposition formulae as follows.

Definition 5 (*Extension of epistemic state*) Let Φ be an epistemic state, then Φ can be extended to any propositional formula μ such that $\Phi(\mu) = \max_{w \models \mu} (\Phi(w))$.

Note that here we use *max* instead of *min*, because in this paper a possible world with a higher value is more important than the one with a lower value whilst for epistemic states in [26] and OCFs, the situation is opposite.

Let $f_\Phi(\mu) = \Phi(\mu) - \Phi(\neg\mu)$, we call $f_\Phi(\mu)$ the *strength of preference* on μ which is interpreted as the relative preference of μ over $\neg\mu$. The notion of strength of preference is not new. In [31], it stated “The strength of preference for a proposition X over

a proposition Y is the expectation (based on an agent's probabilistic beliefs) that a world in X is better than a world in Y ." Our notion follows a similar explanation. For distinction, we call $\Phi(\mu)$ the weight of μ .

Now we consider a special case of epistemic state.

Definition 6 An epistemic state Φ is called a **simple epistemic state** iff $\exists \mu$ such that

$$\Phi(w) = \begin{cases} m & \text{for } w \models \mu, \\ 0 & \text{for } w \not\models \mu. \end{cases}$$

Here m is an integer and we simply write Φ as (μ, m) .

Simple epistemic states are introduced to make the representation of postulates in the next subsection simpler. Note that any epistemic state in which possible worlds can be divided into two sets with two different values assigned can be transformed into semantically equivalent simple epistemic states as discussed above. In classic belief revision, most commonly the new input is a propositional formula μ which could be seen as a special case of simple epistemic state (μ, m) where m is a positive integer.

For a simple epistemic state $\Phi = (\mu, m)$, we have $Bel(\Phi) = \mu$ if $m > 0$; $Bel(\Phi) = \neg\mu$ if $m < 0$, and $Bel(\Phi) = \top$ if $m = 0$. It also shows that if $m = 0$, this simple epistemic state is totally ignorant.

In the next section, for illustration and simplicity, we will first consider using simple epistemic states to represent simple uncertain inputs and then extend to general cases where we use any epistemic states to represent general uncertain inputs.

3.2 Postulates

In this subsection, we propose some rational postulates for belief change by epistemic states and then give representing theorems for these postulates.

Following [19] and [8], we also use the notation $form(w_1, w_2, \dots)$ to denote a proposition μ which has w_1, w_2, \dots as its models, that is, $Mod(\mu) = \{w_1, w_2, \dots\}$. By abuse of notations, we also use $form(A)$ to denote a proposition μ such that $Mod(\mu) = A$. We also assume that when an epistemic state Φ is embedded in a propositional formula, it stands for $Bel(\Phi)$, e.g. $\Phi \wedge \psi$ means $Bel(\Phi) \wedge \psi$; $\Phi \models \mu$ means $Bel(\Phi) \models \mu$ and $Mod(\Phi) = Mod(Bel(\Phi))$, etc.

A belief change operator associating an epistemic state and a new uncertain input (represented in the form of epistemic state) to a resulting epistemic state must satisfy certain constraints formalized as postulates. Now we investigate what postulates should be. Since the key characteristics of belief change is to allow strengths of the current beliefs and new uncertain inputs to determine the outcome, we propose the following seven postulates (with their explanations) to characterize belief change.

B0 $\Phi \circ (\mu, 0) = \Phi$ for any consistent μ .

Explanation: This postulate states that if an agent obtains ignorant information, then its beliefs shall not be changed. Here $(\mu, 0)$ represents a totally ignorant input as discussed above.

B1 If $f_\Phi(\mu) > m$, then $\Phi \circ (\neg\mu, m) \models \mu$; If $f_\Phi(\mu) < m$, then $\Phi \circ (\neg\mu, m) \models \neg\mu$; If $f_\Phi(\mu) = m$, then $\Phi \circ (\neg\mu, m) \not\models \mu$ and $\Phi \circ (\neg\mu, m) \not\models \neg\mu$.

Explanation: This intuitively follows Definition 1, which shows that it should be the strengths of agent's current beliefs and new input that determine the outcome of belief change. When $f_\Phi(\mu) = m$ and new input gives $(\neg\mu, m)$, then neither μ nor $\neg\mu$ should be believed. That is because the prior beliefs prefer μ with strength m whilst the input prefers $\neg\mu$ with the same strength, then no conclusion can be drawn.

B2 If $\Phi \wedge \mu$ is satisfiable, then $\Phi \circ (\mu, m) \equiv \Phi \wedge \mu$ when $m > 0$.

Explanation: If a new input is consistent with an agent's current beliefs, then the agent incorporates the new input into its beliefs.

B3 If μ is satisfiable, then $\Phi \circ (\mu, m)$ is an epistemic state.

Explanation: If a new input is consistent, then a new epistemic state should be obtained after belief change. With this postulate, without loss of generality, we assume that all new inputs considered in this paper are consistent.

B4 If $\Phi_1(w_1) = \Phi_2(w_2)$ and $(\mu_1, m)(w_1) = (\mu_2, n)(w_2)$, then $(\Phi_1 \circ (\mu_1, m))(w_1) = (\Phi_2 \circ (\mu_2, n))(w_2)$.

Explanation: If two agents hold the same (prior) degree of beliefs (strength) on two possible worlds w_1 and w_2 , and two new inputs they receive also have the same strength on these two worlds respectively, then both agents shall still have the same degree of belief on the two worlds after changing their prior beliefs with new inputs. This postulate is intuitively similar to the *Neutrality with respect to the intensity scale* condition proposed in [10] which says in a social choice scenario, an aggregation function should not depend on the semantic meanings of a set of social choice functions, but only focus on their intensities (numerical values in $[0, 1]$) of choices.

B5 $\Phi \circ (\mu, m) \circ (\mu, n) = \Phi \circ (\mu, m + n)$.

Explanation: The strength of beliefs of a proposition is reinforced when multiple new inputs supporting it are received. This postulate requires that these inputs are independently obtained. A typical scenario of this postulate is in a rumor spreading process. Cumulative rumors usually destroy people's current beliefs. This postulate is also in spirit similar to the Dempster's combination rule in the Dempster-Shafer theory of evidence. When the rule is applied to combine consistent distinguished pieces of evidence, it has a reinforcement effect.

B6 $\Phi \circ (\mu, m) \circ (\psi, n) = \Phi \circ (\psi, n) \circ (\mu, m)$.

Explanation: The order of inputs (received) shall not influence the outcome of belief change. This and **B1** together determine the main difference between a belief change operator and belief revision operator.

With these postulates, we have the following results.

Proposition 1 *If a belief change operator \circ satisfies **B0** and **B4**, then we have $(\Phi \circ (\mu, m))(w) = \Phi(w)$ for $w \models \neg\mu$.*

This proposition shows that a new input has no effect on possible worlds which it does not support.

Proposition 2 *If a belief change operator \circ satisfies **B0**, **B1** and **B4**, and if $w \not\models Bel(\Phi)$, then we have*

$$Mod(\Phi \circ (form(w), m)) = \begin{cases} Mod(\Phi) & \text{if } m < f_{\Phi}(\neg form(w)), \\ Mod(\Phi) \cup \{w\} & \text{if } m = f_{\Phi}(\neg form(w)), \\ \{w\} & \text{if } m > f_{\Phi}(\neg form(w)), \end{cases}$$

This proposition states that when the degree of the input supporting w increases, possible worlds w will be gradually becoming a model of the prior belief set and could even becoming the unique model of the final belief set, even though it may not be a model of the prior belief initially.

Now we give the following representation theorem for these postulates.

Theorem 1 *A belief change operator \circ satisfies postulates **B0-B6** precisely when*

$$\forall w, (\Phi \circ (\mu, m))(w) = \Phi(w) + (\mu, m)(w) \quad (2)$$

This theorem³ describes the belief change process when a new input is represented as a simple epistemic state.

This postulate suggests that the operator depicted by the postulates is generally a shifting operator, and so is in line with many existing approaches, e.g., [30, 13, 38, 34, 35, 26, 20, 36, 21, 37, 18], etc. However, Theorem 1 shows that this operator rewards any worlds with the strengths provided by the input, but does not penalize any worlds which are not in the belief set of the input, whilst existing shifting operators will give penalties to the worlds dissatisfying with the input.

From this theorem, we get that if an epistemic state Φ can be represented as:

$$\Phi(w) = m_i \text{ iff } w \models \mu_i, 1 \leq i \leq n,$$

where μ_i s are pairwise contradictory, then we have

$$\forall w, \Phi(w) = ((\mu_1, m_1) \circ \dots \circ (\mu_n, m_n))(w).$$

Now we can extend this theorem to the case that a new input is not restricted to only a simple epistemic state. To accommodate this extension, we need to naturally extend postulate B6 to B6* as follows.

B6* $\Phi_1 \circ \Phi_2 \circ \Phi_3 = \Phi_1 \circ \Phi_3 \circ \Phi_2$.

Then we have the following representation theorem.

³Here we need to point out that $(+\infty) + (-\infty)$ is undefined. It could be any value in $Z \cup \{+\infty, -\infty\}$. Generally, this situation will not occur, since $-\infty$ means absolutely impossible while $+\infty$ means totally sure. Basically a possible world cannot be considered both impossible and sure.

Theorem 2 A belief change operator \circ satisfies postulates **B0-B5** and **B6*** precisely when

$$\forall w, (\Phi \circ \Phi')(w) = \Phi(w) + \Phi'(w). \quad (3)$$

This theorem reveals that with the definition of epistemic states in Definition 2, only the rewarding operator can serve the true purpose of achieving belief change is triggered by the strength of evidence.

Example 2 (Example 1 revisited) Let the prior epistemic state about the initial one-person theory be Φ such that

$$\Phi(\text{John}, \neg \text{Mary}) = \alpha, \Phi(\neg \text{John}, \text{Mary}) = \alpha, \alpha > 0,$$

$$\Phi(\text{John}, \text{Mary}) = 0, \Phi(\neg \text{John}, \neg \text{Mary}) = -\infty.$$

Let the two testimonies incriminating John and Mary be represented respectively by two simple epistemic states as (John, β) and (Mary, γ) where $\beta, \gamma > 0$. Let \circ be a belief change operator on epistemic state Φ satisfying postulates **B0-B6**, then applying \circ to these three epistemic states, we have the final epistemic state Φ_{JM} as

$$\Phi_{JM}(\text{John}, \neg \text{Mary}) = \alpha + \beta, \Phi_{JM}(\neg \text{John}, \text{Mary}) = \alpha + \gamma,$$

$$\Phi_{JM}(\text{John}, \text{Mary}) = \beta + \gamma, \Phi_{JM}(\neg \text{John}, \neg \text{Mary}) = -\infty$$

If the one-person theory is the most reliable evidence, i.e. $\alpha > \max(\beta, \gamma)$, then we have both

$$\Phi_{JM}(\text{John}, \neg \text{Mary}) > \Phi_{JM}(\text{John}, \text{Mary})$$

$$\Phi_{JM}(\neg \text{John}, \text{Mary}) > \Phi_{JM}(\text{John}, \text{Mary})$$

which show that the murderer is one of them which is intuitively what we want to get. Furthermore, who exactly committed the crime is based on the strengths of the evidence β vs. γ . When $\beta > \gamma$, i.e., when $\Phi_{JM}(\text{John}, \neg \text{Mary}) > \Phi_{JM}(\neg \text{John}, \text{Mary})$, then John is the murderer, when $\gamma > \beta$, i.e., when $\Phi_{JM}(\text{John}, \neg \text{Mary}) < \Phi_{JM}(\neg \text{John}, \text{Mary})$, then Mary is the murderer. An interesting situation is when $\beta = \gamma$, we cannot decide who is the murderer and this is intuitively correct also, because the evidence is not against one over the other.

On the other hand, if $\alpha < \min(\beta, \gamma)$, then the result of belief change $(\text{John} \wedge \text{Mary})$ suggests that both John and Mary are murderers which is also intuitively explainable.

This example shows that belief change operators satisfying postulates **B0-B6** do allow the strengths of evidence play an essential role in determining the outcome of belief change.

3.3 A justification on OCF combination

Below we prove that our definition of epistemic state and the belief change rule $(\Phi \circ \Phi')(w) = \Phi(w) + \Phi'(w)$ given in Theorem 2 can induce the ordinal conditional function and its combination method defined by Equation 1, respectively. Thus, our postulates justify the rationale for the combination of ordinal conditional functions using Equation 1.

Definition 7 Let Φ be an epistemic state defined in Definition 2. We define $\kappa_\Phi : \mathcal{L} \rightarrow Z$ as a corresponding function for Φ such that

$$\kappa_\Phi(\mu) = \max_{w \in W} (\Phi(w)) - \Phi(\mu). \quad (4)$$

For $(+\infty) - (+\infty)$ and $(-\infty) - (-\infty)$, the results are undefined as explained early. We have the following immediate result.

Proposition 3 Let Φ be an epistemic state and κ_Φ be its corresponding function based on Definition 7, then κ_Φ is an ordinal conditional function.

The following theorem shows that the result of changing an epistemic state Φ with an uncertain input represented by Φ' is equivalent to the result of combining the two corresponding functions derived from Φ and Φ' respectively.

Theorem 3 Let Φ and Φ' be two epistemic states and $\Phi \circ \Phi'$ be the resulting epistemic state after belief change. Let κ_Φ , $\kappa_{\Phi'}$, and $\kappa_{\Phi \circ \Phi'}$ be their corresponding functions respectively, then we have $\forall w, \kappa_{\Phi \circ \Phi'}(w) = (\kappa_\Phi \oplus \kappa_{\Phi'})(w)$ where \oplus is the combination operator in Equation 1.

The correspondence between epistemic states and OCFs given by Theorem 3 is intuitively illustrated as follows.

$$\begin{array}{ccc} (\Phi \circ \Phi')(w) & = & \Phi(w) + \Phi'(w) \\ \downarrow & & \downarrow \quad \downarrow \\ \kappa_{\Phi \circ \Phi'}(w) & = & (\kappa_\Phi \oplus \kappa_{\Phi'})(w) \end{array}$$

Example 3 A father believed that his child X is clever (c) and very honest (h). The prior epistemic state Φ can then be constructed as

$$\Phi(c, h) = 15, \Phi(\neg c, h) = 12, \Phi(c, \neg h) = 5, \Phi(\neg c, \neg h) = 0.$$

Now someone told him that X told a lie. This new input can be captured by a simple epistemic state $(\neg h, 2)$. Applying the belief change operator \circ to Φ with new input $(\neg h, 2)$, the father obtained his revised epistemic state, $\Phi_{\neg h} = \Phi \circ (\neg h, 2)$, as

$$\Phi_{\neg h}(c, h) = 15, \Phi_{\neg h}(\neg c, h) = 12, \Phi_{\neg h}(c, \neg h) = 7, \Phi_{\neg h}(\neg c, \neg h) = 2,$$

which shows that although he still believes that his child is largely honest, however his belief in the child's dishonesty has been increased.

Based on Definition 7, the corresponding κ functions for Φ , $(\neg h, 2)$, and $\Phi_{\neg h}$ can be obtained as follows.

$$\kappa_{\Phi}(c, h) = 0, \kappa_{\Phi}(\neg c, h) = 3, \kappa_{\Phi}(c, \neg h) = 10, \kappa_{\Phi}(\neg c, \neg h) = 15.$$

$$\kappa_{\neg h}(h) = 2, \kappa_{\neg h}(\neg h) = 0.$$

$$\kappa_{\Phi_{\neg h}}(c, h) = 0, \kappa_{\Phi_{\neg h}}(\neg c, h) = 3.$$

$$\kappa_{\Phi_{\neg h}}(c, \neg h) = 8, \kappa_{\Phi_{\neg h}}(\neg c, \neg h) = 13.$$

It is easy to verify that $\kappa_{\Phi_{\neg h}}(w) = \kappa_{\Phi}(w) + \kappa_{\neg h}(w) - \min_{w \in W} (\kappa_{\Phi}(w) + \kappa_{\neg h}(w))$ for any possible world w .

4 Belief Change vs. Belief Revision

4.1 Darwiche and Pearl's postulates on iterated belief revision

To demonstrate the inadequacy of iterated belief revision, in [8] Darwiche and Pearl deployed a set of examples to show how counterintuitive results will appear if AGM postulates are to be followed. They recommended that to ensure the rational preservation of conditional beliefs⁴ during (iterated) belief revision, a revision process shall be carried out on epistemic states⁵ rather than on their belief sets. With this intention, epistemic states are used to represent an agent's original beliefs and a new inputs which is taken as a propositional formula. Correspondingly, Darwiche and Pearl modified the AGM postulates (rephrased by Katsuno and Mendelzon in [19] as R1-R6) to obtain a set of revised postulates (R*1-R*6) for iterated epistemic revision. Let \circ_r be a revision operator which revises an epistemic state with a propositional formula to a new epistemic state, the revised postulates are

R*1 $\Psi \circ_r \mu$ implies μ .

R*2 If $\Psi \wedge \mu$ is satisfiable, then $\Psi \circ_r \mu \equiv \Psi \wedge \mu$.

R*3 If μ is satisfiable, then $\Psi \circ_r \mu$ is also satisfiable.

R*4 If $\Psi_1 = \Psi_2$ and $\mu_1 \equiv \mu_2$, then $\Psi_1 \circ_r \mu_1 \equiv \Psi_2 \circ_r \mu_2$.

R*5 $(\Psi \circ_r \mu) \wedge \phi$ implies $\Psi \circ_r (\mu \wedge \phi)$.

R*6 If $(\Psi \circ_r \mu) \wedge \phi$ is satisfiable, then $\Psi \circ_r (\mu \wedge \phi)$ implies $(\Psi \circ_r \mu) \wedge \phi$.

In the above postulates, Ψ (or Ψ_1, Ψ_2) stands for an epistemic state and μ and ϕ are propositional formulae. $\Psi \circ_r \mu$ is an epistemic state after revising Ψ with revision operator \circ_r by μ . When an epistemic state (e.g., Ψ) is embedded in a propositional

⁴Again, in this section, the definition of epistemic state follows DP's setting as mentioned in footnote 2.

⁵Let ψ and α be two propositional formulae and let \circ_r be a belief revision operator, then the revision of ψ by α is a new propositional formula and is denoted as $\psi \circ_r \alpha$. $\beta|\alpha$ is called a conditional belief of ψ if $\psi \circ_r \alpha \models \beta$ [8].

formula, it is used to stand for its belief set⁶ (e.g., $Bel(\Psi)$) not an epistemic state for simplification purpose. For example, $\Psi \wedge \phi$ means $Bel(\Psi) \wedge \phi$.

To regulate iterated epistemic revision to preserve conditional beliefs, Darwiche and Pearl gave the following four additional postulates which are for four disjoint types of conditional beliefs.

C1 If $\alpha \models \mu$, then $(\Psi \circ_r \mu) \circ_r \alpha \equiv \Psi \circ_r \alpha$.

C2 If $\alpha \models \neg\mu$, then $(\Psi \circ_r \mu) \circ_r \alpha \equiv \Psi \circ_r \alpha$.

C3 If $\Psi \circ_r \alpha \models \mu$, then $(\Psi \circ_r \mu) \circ_r \alpha \models \mu$.

C4 If $\Psi \circ_r \alpha \not\models \neg\mu$, then $(\Psi \circ_r \mu) \circ_r \alpha \not\models \neg\mu$.

$\Psi \circ_r \alpha \models \beta$ here stands for $Bel(\Psi \circ_r \alpha) \models \beta$.

Two representation theorems are given to characterize these two sets of postulates, which we call DP postulates in the rest of the paper. But first, we introduce the definition of faithful assignment.

Definition 8 [8] Let W be the set of all worlds (interpretations) of a propositional language \mathcal{L} and suppose that the belief set of any epistemic state belongs to \mathcal{L} . A function that maps each epistemic state Φ to a total pre-order \leq_Φ^r on worlds W is said to be a faithful assignment if and only if:

1. $w_1, w_2 \models \Phi$ only if $w_1 =_\Phi^r w_2$.
2. $w_1 \models \Phi$ and $w_2 \not\models \Phi$ only if $w_1 <_\Phi^r w_2$.
3. $\Phi = \Psi$ only if $\leq_\Phi^r = \leq_\Psi^r$.

Theorem 4 [8] A revision operator \circ_r satisfies postulates **R*1-R*6** precisely when there exists a faithful assignment that maps each epistemic state Φ to a total pre-order \leq_Φ^r such that:

$$Mod(\Phi \circ_r \mu) = \min(Mod(\mu), \leq_\Phi^r).$$

This representation theorem shows that the revised belief is determined by μ and the total pre-order associated with Φ .

Theorem 5 [8] Suppose that a revision operator \circ_r satisfies postulates **R*1-R*6**. The operator satisfies postulates **C1-C4** iff the operator and its corresponding faithful assignment satisfy:

CR1 If $w_1 \models \mu$ and $w_2 \models \mu$, then $w_1 \leq_\Phi^r w_2$ iff $w_1 \leq_{\Phi \circ_r \mu}^r w_2$.

CR2 If $w_1 \models \neg\mu$ and $w_2 \models \neg\mu$, then $w_1 \leq_\Phi^r w_2$ iff $w_1 \leq_{\Phi \circ_r \mu}^r w_2$.

⁶In [8], it demonstrated that each epistemic state has an associated belief set which characterizes the set of propositions that the agent is committed to at any given time.

CR3 If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 <_{\Phi}^r w_2$ only if $w_1 <_{\Phi \circ_r \mu}^r w_2$.

CR4 If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 \leq_{\Phi}^r w_2$ only if $w_1 \leq_{\Phi \circ_r \mu}^r w_2$.

This representation theorem states that an epistemic revision operator \circ_r satisfies postulate Ci iff condition CRi is satisfied, $1 \leq i \leq 4$.

4.2 Belief change versus belief revision

In this section, we want to prove that when reducing to the belief revision situation, our result should derive all the belief revision postulates including Darwiche and Pearl's belief revision postulates R*1-R*6 and C1-C4 [8]. Because the essential difference between belief change and belief revision is that belief revision takes the most recent input as the most reliable one. From our view of belief change, belief revision always assigns a reasonably larger strength of preference to the most recent input. For instance, given a prior epistemic state Φ , it is possible to create a new input (μ, m^*) such that $m^* = \max_{w \in W}(\Phi(w)) - \min_{w \in W}(\Phi(w)) + 1$, then the strength of new input μ is stronger than the strength of belief on any subset A of 2^W . To simulate an iterated belief revision process with evidence sequence μ_1, \dots, μ_n using a belief change operator, we only need to let $m_1 = m^*$, and $m_i = 2^{i-1} * m_1$, $1 < i \leq n$ to ensure that each new input always has a stronger strength than any previous inputs. For convenience, we name (μ, m) **revision input** if $m \geq m^*$. Intuitively, a revision input means that the strength of new input is large enough to make the information implied in the input being influential to the outcome of the revision.

Theorem 6 Let Φ be an epistemic state and (μ, m) be a revision input, then a belief change operator \circ satisfying postulates **B0-B6** guarantees the existence of a faithful assignment which maps each epistemic state Φ to a total pre-order \preceq_{Φ} such that:

$$Mod(\Phi \circ (\mu, m)) = \min(Mod(\mu), \preceq_{\Phi})$$

This theorem, accompanied with Theorem 4, shows that the belief set obtained from belief change on an epistemic state Φ with revision input (μ, m) is equivalent to the belief set obtained from belief revision on Φ with formula μ . That is $Bel(\Phi \circ (\mu, m)) = Bel(\Phi \circ_r \mu)$.

Theorem 7 Let Φ be an epistemic state and (μ, m) be a revision input, a belief change operator \circ satisfies postulates **B0-B6** implies the following:

CR1* If $w_1 \models \mu$ and $w_2 \models \mu$, then $w_1 \preceq_{\Phi} w_2$ iff $w_1 \preceq_{\Phi \circ (\mu, m)} w_2$.

CR2* If $w_1 \models \neg\mu$ and $w_2 \models \neg\mu$, then $w_1 \preceq_{\Phi} w_2$ iff $w_1 \preceq_{\Phi \circ (\mu, m)} w_2$.

CR3* If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 <_{\Phi} w_2$ only if $w_1 <_{\Phi \circ (\mu, m)} w_2$.

CR4* If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 \leq_{\Phi} w_2$ only if $w_1 \leq_{\Phi \circ (\mu, m)} w_2$.

CRI* corresponds to CRI, $1 \leq i \leq 4$. This theorem reveals that our belief change postulates subsume Darwiche-Pearl's iterated belief revision postulates C1-C4. Theorem 6 and Theorem 7 together show that our set of postulates is indeed an extension of iterated belief revision when equipped with strengths.

We can also prove that our postulates induce the following Recalcitrance (Rec) postulate [29] and Independence (Ind) postulate [18] when reduced to belief revision situation.

Rec If $\alpha \not\models \neg\mu$, then $(\Phi \circ_r \mu) \circ_r \alpha \models \mu$.

Ind If $\Phi \circ_r \neg\alpha \not\models \neg\mu$, then $(\Phi \circ_r \mu) \circ_r \neg\alpha \models \mu$.

Semantically, given \circ_r satisfying **R*1-R*6**, postulate **Rec** and **Ind** correspond to the following conditions [29, 18].

RecR If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 <_{\Phi \circ_r \mu}^r w_2$.

IndR If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 \leq_{\Phi}^r w_2$ only if $w_1 <_{\Phi \circ_r \mu}^r w_2$.

The following theorem shows our postulates truly induce the Recalcitrance postulate and the Independence postulate.

Theorem 8 *Let Φ be an epistemic state and (μ, m) be a revision input, a belief change operator \circ satisfying postulates **B0-B6** implies the following:*

RecR* *If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 \prec_{\Phi \circ(\mu, m)} w_2$.*

IndR* *If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 \preceq_{\Phi} w_2$ only if $w_1 \prec_{\Phi \circ(\mu, m)} w_2$.*

This is not surprising as the Recalcitrance postulate implies the Independence postulate which in turn implies C3 and C4.

In summary, if the strength of new information is large enough (hence such information is a piece of revision evidence), then the belief change operator reduces to a belief revision operator. This truly reflects *the strength of evidence triggers the change*.

5 Related Work

In this section, we discuss and compare with some related work.

5.1 Iterated conditional revision

In [21], epistemic states are defined as logic formulae equipped with a numerical data structure called the *conditional valuation function* which generalizes OCFs and probability functions, and then iterated conditional belief revision is axiomatized and investigated. Iterated conditional revision has also been studied by Weydert within the ranking measure framework [35, 37] in which epistemic states are represented as ranking measures [33] which take the ordinal conditional functions as a special case. Both Kern-Isberner's and Weydert's work characterize and elaborate approaches to iterated revision that are also shifting approaches similar to the approach proposed in this paper (i.e., Theorem 1). However, both of the above two frameworks still fall into the standard revision protocol in which the success postulate is followed.

5.2 Belief merging - Konieczny and Pino-Pérez postulates

In [22], a set of postulates on the merging of flat knowledge bases was presented. A (*flat*) *knowledge base* K is a finite set of propositions. K is consistent iff there is at least one interpretation that satisfies all propositions in K .

A knowledge profile E is a multi-set of knowledge bases such that $E = \{K_1, K_2, \dots, K_n\}$ where K_i , $1 \leq i \leq n$, is a knowledge base. $\sqcup E = K_1 \sqcup \dots \sqcup K_n$ denotes the set union of K_i s and $\bigwedge E = K_1 \wedge \dots \wedge K_n$ denotes the conjunction of knowledge bases K_i s of E . E is called consistent iff $\bigwedge E$ is consistent. $E_1 \leftrightarrow E_2$ denotes that there is a bijection g from $E_1 = \{K_1^1, \dots, K_n^1\}$ to $E_2 = \{K_1^2, \dots, K_n^2\}$ such that $\vdash g(K) \leftrightarrow K$.

An operator Δ is a mapping from knowledge profiles to knowledge bases. And Δ is a merging operator iff it satisfies the following postulates.

A1 $\Delta(E)$ is consistent.

A2 If E is consistent, then $\Delta(E) = \bigwedge E$.

A3 If $E_1 \leftrightarrow E_2$, then $\vdash \Delta(E_1) \leftrightarrow \Delta(E_2)$.

A4 If $K \wedge K'$ is not consistent, then $\Delta(K \sqcup K') \not\vdash K$.

A5 $\Delta(E_1) \wedge \Delta(E_2) \vdash \Delta(E_1 \sqcup E_2)$.

A6 If $\Delta(E_1) \wedge \Delta(E_2)$ is consistent, then $\Delta(E_1 \sqcup E_2) \vdash \Delta(E_1) \wedge \Delta(E_2)$.

According to [22], A1-A6 can be interpreted as follows. A1 says that we always want to extract some information from the knowledge profile. A2 assures that if all the knowledge bases are consistent, then the consistent information will be the merging result. A3 implies merging is irrelevant of syntax. A4 demonstrates that the merging is fair. A5 and A6 together states that if we could find two subgroups which agree on at least one alternative, then the result of global arbitration will be exactly those alternatives the two subgroups agree on.

Evidently, we can see that A1 is similar to B3, A2 is similar to B2. Since our postulates are on epistemic states which are attached with strengths, there is no direct counterpart of A3, but B4 can somehow be seen as a quantitative extension of A3. B1 is a refined version of A4. That is, in a pure logic-based framework, it is only possible to determine that K and K' are inconsistent, but it is normally impossible to quantify to what degree they are inconsistent⁷. So A4 only says that no preference should be given to either of the knowledge bases if they are inconsistent, which is questioned by many researchers. However, in our framework, strengths of prior beliefs and new inputs serve as an elaborated tool to tell explicitly the degree of inconsistency, so it is possible for us to define a much more refined postulate B1 than postulate A4. Note that in A5 and A6, E_1 and E_2 are two knowledge profiles, each containing a set of knowledge bases, hence, it is possible to perform the merging of each knowledge profile first (i.e., $\Delta(E_1)$, $\Delta(E_2)$) and then their merging results can be finally merged. However, in

⁷Although there are measures of inconsistency of a single knowledge base [16, 27], there are no standard measures about the degree of inconsistency between two knowledge bases.

our framework, both the prior beliefs and the new input are in fact represented as an individual epistemic state, hence no such extra actions could be taken for managing them, prior to their merging. Therefore, we cannot define extensions for A5 and A6 in our framework. However, there is a common gesture between belief merging and our belief change framework, that is the *step by step* merging implied by A5 and A6, which, to some extent, is exhibited by B5 and B6 about step by step belief change.

In [26], epistemic state merging properties are studied where epistemic states are in fact OCFs. Since these properties are direct quantitative counterparts of the above Konieczny and Pino-Pérez postulates, here we omit the detailed comparison with these properties. Remarkably, an operator similar to the rewarding operator in Theorem 2 is also mentioned as an example of epistemic state merging operator in [26].

5.3 Multiple belief change

Multiple belief change, including multiple contraction [12, 39] and multiple revision [39], is summarized in [40] in which these two approaches are mutually transformable. Multiple belief revision considers situations that more than one input is obtained simultaneously and hence all these inputs are used for revision.

Postulate B6 seems to imply that our framework entails a multiple revision process. However, there are some significant differences. The major difference is that multiple revision does not consider strengths. In addition, new inputs are represented by sentences in multiple revision whilst in our framework, all information is represented by epistemic states. Furthermore, multiple belief change is still a *revision* such that a new input should be respected while our framework does not follow this.

5.4 Non-Prioritized belief revision

Our belief change framework can be seen as closely related to the non-prioritized belief revision (cf. [14] for an overview). However, similar to the multiple revision discussed previously, strength does not play a role in non-prioritized belief revision. In addition, non-prioritized revision is a sentence-based revision process which is different from our epistemic state based change process.

5.5 Revision as prioritized merging

In [9], it was proposed that belief revision could be considered as prioritized merging. Essentially, [9] still studies the standard revision strategy that the success postulates is retained, but the revision strategy was viewed from a merging perspective. In [9], strengths were not considered in revision and merging. However, the necessity of using strengths in revision and merging to achieve more complex objectives was discussed.

5.6 Iterated revision on epistemic states

In [25], an iterated revision framework on epistemic states is proposed in which a more general definition on epistemic states was defined which takes the definition of epistemic states in this paper as a special case. Although strengths are introduced in the

definition of epistemic states in [25], the basic framework is still performing revision in nature which respects the new input whilst the belief change framework proposed in this paper does not. The commonality of the two papers is that new inputs should be represented by epistemic states, not logical formulae.

6 Conclusion

In this paper, we presented a framework for managing belief change with uncertain inputs, in which prior beliefs and inputs are both represented in the form of epistemic states. We proposed a set of postulates to characterize belief change operators. We also provided representation theorems for our postulates which show that our belief change operators are in fact rewarding operators. In addition, we show that these postulates can be seen as a justification of the combination of two OCFs [23].

When reducing to the iterated belief revision situation by Darwiche and Pearl (where a new input is a propositional formula), our postulates induce all of DP’s postulates. Moreover, we also proved that the Recalcitrance postulate in [29] and the Independence postulate in [18] can be induced from our postulates as well.

Belief revision requires that the most recent evidence is treated as the most reliable one. When it is not the case, belief revision cannot be applied, as studied in [7, 14, 6, 9]. However, these approaches cannot achieve the goal that *the outcome of belief change depends on the strength of evidence triggering the change*⁸ [8]. Our approach, inspired by this observation, tackles this problem by deploying epistemic states to model new inputs and their strengths, so that the strengths of evidence *trigger* belief change.

As Kern-Isberner’s [21] and Weydert’s [35, 37] work also deal with *shifting* iterated revision, it is an interesting future work to compare the axioms used in these frameworks to the axioms used in this paper.

Postulate **B4** is somehow a rather strong postulate which restricts the combination of epistemic states to be point-wise combination. Further research on how to weaken this postulate would be interesting and worthwhile.

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⁸That is one of the reasons why this paper uses the phrase *belief change* in the title.

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Appendix

Proof of Proposition 1: By setting $\Phi_1 = \Phi_2 = \Phi$, $w_1 = w_2 = w$, $(\mu_1, m) = (\mu, m)$ and $(\mu_2, n) = (\mu, 0)$, from **B4**, we get $(\Phi \circ (\mu, m))(w) = (\Phi \circ (\mu, 0))(w)$. From **B0**, we have $(\Phi \circ (\mu, 0))(w) = \Phi(w)$, thus $(\Phi \circ (\mu, m))(w) = \Phi(w)$.

Proof of Proposition 2: Let $\mu = form(w)$ and $w' \models \Phi$, then we have

$$\begin{aligned}
 f_\Phi(\neg\mu) &= \Phi(\neg\mu) - \Phi(\mu) \\
 &= \max_{w^* \models \neg\mu} (\Phi(w^*)) - \max_{w^* \models \mu} (\Phi(w^*)) \\
 &= \max_{w^* \neq w} (\Phi(w^*)) - \Phi(w) \\
 &= \Phi(w') - \Phi(w)
 \end{aligned}$$

As \circ satisfies **B0** and **B4**, then from Proposition 1, we know that $\forall w^* \neq w$, $(\Phi \circ (form(w), m))(w^*) = \Phi(w^*)$. Thus only the epistemic strength of w is changed, and $Mod(\Phi \circ (form(w), m))$ depends on the value $(\Phi \circ (form(w), m))(w)$.

From **B1**, we have

1. If $m < \Phi(w') - \Phi(w) = f_\Phi(\neg\mu)$, then $\Phi \circ (form(w), m) \models \neg form(w)$, hence $w \not\models (\Phi \circ (form(w), m))$. Then $Mod(\Phi \circ (form(w), m)) = Mod(\Phi)$.
2. If $m = \Phi(w') - \Phi(w) = f_\Phi(\neg\mu)$, then $\Phi \circ (form(w), m) \not\models \neg form(w)$ implies $w \models (\Phi \circ (form(w), m))$, and $\Phi \circ (form(w), m) \not\models form(w)$ implies that there exists some $w^* \neq w$ such that $w^* \models (\Phi \circ (form(w), m))$. Recall $(\Phi \circ (form(w), m))(w^*) = \Phi(w^*)$ as $w^* \neq w$, we get $Mod(\Phi \circ (form(w), m)) = Mod(\Phi) \cup \{w\}$.
3. If $m > \Phi(w') - \Phi(w) = f_\Phi(\neg\mu)$, then $\Phi \circ (form(w), m) \models form(w)$, hence w is the only model of $\Phi \circ (form(w), m)$. Then $Mod(\Phi \circ (form(w), m)) = \{w\}$.

Proof of Theorem 1: (\Rightarrow) Suppose there is a belief change operator \circ satisfies postulates **B0-B6**. We will show $(\Phi \circ (\mu, m))(w) = \Phi(w) + (\mu, m)(w)$ holds by the following steps.

Step 1, we show that $(\Phi \circ (form(w), m))(w) \geq \Phi(w) + m$ if $w \in Mod(\Phi)$ and $m \geq 0$.

The situation $m = 0$ is straightforward by **B0**. For $m \geq 1$, first we prove that $(\Phi \circ (form(w), 1))(w) \geq \Phi(w) + 1$. We consider the following two cases.

1. $w \in Mod(\Phi)$ and there exists another possible world $w' \in Mod(\Phi)$.
As $w \in Mod(\Phi)$, then $\Phi \wedge form(w)$ is satisfiable (w is its unique model), hence from **B2**, we have $\Phi \circ (form(w), 1) \equiv \Phi \wedge form(w)$. So we get that $Mod(\Phi \circ (form(w), 1)) = \{w\}$. Then we have $(\Phi \circ (form(w), 1))(w) > (\Phi \circ (form(w), 1))(w')$. From Proposition 1, we get $(\Phi \circ (form(w), 1))(w') = \Phi(w')$. As both $w, w' \in Mod(\Phi)$, we have $\Phi(w) = \Phi(w')$. So finally we have $(\Phi \circ (form(w), 1))(w) > \Phi(w)$, hence we have $(\Phi \circ (form(w), 1))(w) \geq \Phi(w) + 1$.
2. $Mod(\Phi) = \{w\}$.
Let $\mu = \neg form(w)$, and $t = f_\Phi(form(w)) > 0$, then from **B1**, we have $\Phi \circ (\mu, t) \not\models \mu$ and $\Phi \circ (\mu, t) \not\models \neg \mu$. $\Phi \circ (\mu, t) \not\models \mu$ implies that $Mod(\Phi \circ (\mu, t)) \cap Mod(\neg \mu) \neq \emptyset$, but $Mod(\neg \mu) = Mod(form(w)) = \{w\}$, so it should be $Mod(\Phi \circ (\mu, t)) \cap Mod(\neg \mu) = \{w\}$, and hence $w \in Mod(\Phi \circ (\mu, t))$. Similarly, $\Phi \circ (\mu, t) \not\models \neg \mu$ implies that there exists some $w' \in Mod(\Phi \circ (\mu, t))$ and $w' \neq w$. Hence a similar proof as case 1 gives that $(\Phi \circ (\mu, t) \circ (form(w), 1))(w) \geq (\Phi \circ (\mu, t))(w) + 1$. From **B6**, we have $(\Phi \circ (\mu, t) \circ (form(w), 1))(w) = (\Phi \circ (form(w), 1) \circ (\mu, t))(w)$. As $w \models \neg \mu$, from Proposition 1, we have $(\Phi \circ (form(w), 1) \circ (\mu, t))(w) = (\Phi \circ (form(w), 1))(w)$ and $(\Phi \circ (\mu, t))(w) = \Phi(w)$. So finally we have $(\Phi \circ (form(w), 1))(w) \geq \Phi(w) + 1$.

From the above we conclude that $(\Phi \circ (form(w), 1))(w) \geq \Phi(w) + 1$ if $w \in Mod(\Phi)$. And from **B2**, we have $Mod(\Phi \circ (form(w), 1)) = \{w\}$, hence we get $(\Phi \circ (form(w), 1) \circ (form(w), 1))(w) \geq (\Phi \circ (form(w), 1))(w) + 1 \geq \Phi(w) + 2$. From **B5**, we have $(\Phi \circ (form(w), 1) \circ (form(w), 1))(w) = (\Phi \circ (form(w), 2))(w)$. So we get $(\Phi \circ (form(w), 2))(w) \geq \Phi(w) + 2$. Similarly, by induction, finally we get $(\Phi \circ (form(w), m))(w) \geq \Phi(w) + m$.

Step 2, we show that for $w \notin Mod(\Phi)$, let $m_w = f_\Phi(\neg form(w)) > 0$, then $(\Phi \circ (form(w), i))(w) = \Phi(w) + i$ for $0 \leq i \leq m_w$.

First we prove that $(\Phi \circ (form(w), m_w)) = \Phi(w) + m_w$ for $w \notin Mod(\Phi)$.

Let $\mu = form(w)$, as $w \notin Mod(\Phi)$, it is easy to see that $m_w = f_\Phi(\neg \mu) > 0$. From Proposition 2, we have $Mod(\Phi \circ (\mu, m_w)) = Mod(\Phi) \cup \{w\}$. Let $w' \in Mod(\Phi)$, then w, w' are both models of $\Phi \circ (\mu, m_w)$, we have $(\Phi \circ (\mu, m_w))(w') = (\Phi \circ (\mu, m_w))(w)$. In addition, from Proposition 1, we have $\Phi(w') = \Phi \circ (\mu, m_w)(w')$, thus

$$\begin{aligned} m_w &= f_\Phi(\neg \mu) = \Phi(\neg \mu) - \Phi(\mu) \\ &= \max_{w^* \models \neg \mu} (\Phi(w^*)) - \Phi(w) \\ &= \Phi(w') - \Phi(w) \end{aligned}$$

$$\begin{aligned}
&= (\Phi \circ (\mu, m_w))(w') - \Phi(w) \\
&= (\Phi \circ (\mu, m_w))(w) - \Phi(w).
\end{aligned}$$

That is $(\Phi \circ (\mu, m_w))(w) = \Phi(w) + m_w$. And we also have $\Phi(w') = \Phi(w) + m_w$.

Now for $0 \leq i < m_w$, we show $(\Phi \circ (\mu, i))(w) = \Phi(w) + i$. The situation that $i = 0$ follows directly from **B0**. For situations $0 < i < m_w$, from Proposition 2, we have $Mod(\Phi \circ (\mu, i)) = Mod(\Phi)$, so $w \notin Mod(\Phi \circ (\mu, i))$. Let $m_i = f_{\Phi \circ (\mu, i)}(\neg\mu)$, similar to the case of m_w , we should have $(\Phi \circ (\mu, i) \circ (\mu, m_i))(w) = (\Phi \circ (\mu, i))(w) + m_i$.

From **B5** and since

$$\begin{aligned}
m_i &= f_{\Phi \circ (\mu, i)}(\neg\mu) \\
&= (\Phi \circ (\mu, i))(\neg\mu) - (\Phi \circ (\mu, i))(\mu) \\
&= (\Phi \circ (\mu, i))(w') - (\Phi \circ (\mu, i))(w) \\
&= \Phi(w) + m_w - (\Phi \circ (\mu, i))(w),
\end{aligned}$$

we get

$$\begin{aligned}
&(\Phi \circ (\mu, i + \Phi(w) + m_w - (\Phi \circ (\mu, i))(w)))(w) \\
&= (\Phi \circ (\mu, i + m_i))(w) \\
&= (\Phi \circ (\mu, i))(w) + m_i \\
&= \Phi(w) + m_w.
\end{aligned}$$

If $(\Phi \circ (\mu, i))(w) < \Phi(w) + i$, then $i + \Phi(w) + m_w - (\Phi \circ (\mu, i))(w) > m_w$, or we can write $i + \Phi(w) + m_w - (\Phi \circ (\mu, i))(w) = m_w + \alpha$ where $\alpha > 0$. Thus we should have $\Phi(w) + m_w = (\Phi \circ (\mu, m_w + \alpha))(w) = (\Phi \circ (\mu, m_w) \circ (\mu, \alpha))(w)$. But since $w \in Mod(\Phi \circ (\mu, m_w))$ and **B2**, we have $Mod(\Phi \circ (\mu, m_w) \circ (\mu, \alpha)) = \{w\}$ which implies that $(\Phi \circ (\mu, m_w) \circ (\mu, \alpha))(w) > (\Phi \circ (\mu, m_w) \circ (\mu, \alpha))(w') = \Phi(w') = \Phi(w) + m_w$. It leads to a contradiction.

If $(\Phi \circ (\mu, i))(w) > \Phi(w) + i$, then $i + \Phi(w) + m_w - (\Phi \circ (\mu, i))(w) < m_w$, or we can write $i + \Phi(w) + m_w - (\Phi \circ (\mu, i))(w) = m_w - \alpha$ where $\alpha > 0$. Thus we should have $\Phi(w) + m_w = (\Phi \circ (\mu, m_w - \alpha))(w)$. But as $m_w - \alpha < m_w$, from Proposition 2, we have $w \notin Mod(\Phi \circ (\mu, m_w - \alpha)) = Mod(\Phi)$ while $w' \in Mod(\Phi \circ (\mu, m_w - \alpha)) = Mod(\Phi)$, so we get $(\Phi \circ (\mu, m_w - \alpha))(w) < (\Phi \circ (\mu, m_w - \alpha))(w') = \Phi(w') = \Phi(w) + m_w$ which leads to a contradiction.

So finally we conclude that it should be $(\Phi \circ (\mu, i))(w) = \Phi(w) + i$.

Step 3, we show that for $w \notin Mod(\Phi)$ and $\forall n > m_w = f_\Phi(\neg form(w))$, $(\Phi \circ (form(w), n))(w) = \Phi(w) + n$ still holds.

Following the proof in Step 2, we still let $w' \in Mod(\Phi)$. From **B6**, we have $\Phi \circ (form(w'), n) \circ (form(w), n) = \Phi \circ (form(w), n) \circ (form(w'), n)$.

From **B2**, we know that $Mod(\Phi \circ (form(w'), n)) = \{w'\}$, and in Step 1, we have shown that $(\Phi \circ (form(w'), n))(w) \geq \Phi(w') + n$. Thus we have

$$\begin{aligned}
&f_{\Phi \circ (form(w'), n)}(\neg form(w)) \\
&= (\Phi \circ (form(w'), n))(\neg form(w)) - (\Phi \circ (form(w'), n))(form(w)) \\
&= (\Phi \circ (form(w'), n))(w') - (\Phi \circ (form(w'), n))(w)
\end{aligned}$$

$$\begin{aligned}
&\geq \Phi(w') + n - \Phi(w) \\
&> n,
\end{aligned}$$

so by the conclusion of Step 2, we have $(\Phi \circ (\text{form}(w'), n) \circ (\text{form}(w), n))(w) = (\Phi \circ (\text{form}(w'), n))(w) + n$. But from Proposition 1, we have $(\Phi \circ (\text{form}(w'), n))(w) = \Phi(w)$, hence we get $(\Phi \circ (\text{form}(w'), n) \circ (\text{form}(w), n))(w) = \Phi(w) + n$. But $(\Phi \circ (\text{form}(w'), n) \circ (\text{form}(w), n))(w) = (\Phi \circ (\text{form}(w), n) \circ (\text{form}(w'), n))(w)$, thus we get $(\Phi \circ (\text{form}(w), n) \circ (\text{form}(w'), n))(w) = \Phi(w) + n$. Still by Proposition 1, we have $(\Phi \circ (\text{form}(w), n) \circ (\text{form}(w'), n))(w) = (\Phi \circ (\text{form}(w), n))(w)$. So finally we have $(\Phi \circ (\text{form}(w), n))(w) = \Phi(w) + n$.

Step 4, we show that for any $w' \in \text{Mod}(\Phi)$, we also have $(\Phi \circ (\text{form}(w'), m))(w) = \Phi(w') + m$ for $m \geq 0$.

We only need to show that $(\Phi \circ (\text{form}(w'), m))(w) = \Phi(w') + m$ for $m > 0$ as the situation $m = 0$ is straightforward from **B0**. Let $w \notin \text{Mod}(\Phi)$ and let $m^* = m + f_\Phi(\neg \text{form}(w)) = m + \Phi(w') - \Phi(w)$.

From **B6**, we have $\Phi \circ (\text{form}(w'), m) \circ (\text{form}(w), m^*) = \Phi \circ (\text{form}(w), m^*) \circ (\text{form}(w'), m)$.

From **B5**, $\Phi \circ (\text{form}(w), m^*) = \Phi \circ (\text{form}(w), f_\Phi(\neg \text{form}(w))) \circ (\text{form}(w), m)$, then from Proposition 2 and **B2**, we have $\text{Mod}(\Phi \circ (\text{form}(w), m^*)) = \{w\}$. As showed in Step 2 and Step 3, we have $(\Phi \circ (\text{form}(w), m^*))(w) = \Phi(w) + m^*$ and from Proposition 1, we have $(\Phi \circ (\text{form}(w), m^*))(w') = \Phi(w')$. Thus we get

$$\begin{aligned}
&f_{\Phi \circ (\text{form}(w), m^*)}(\neg \text{form}(w')) \\
&= (\Phi \circ (\text{form}(w), m^*))(\neg \text{form}(w')) - (\Phi \circ (\text{form}(w), m^*))(\text{form}(w')) \\
&= (\Phi \circ (\text{form}(w), m^*))(w) - (\Phi \circ (\text{form}(w), m^*))(w') \\
&= \Phi(w) + m^* - \Phi(w') \\
&= \Phi(w) + m + \Phi(w') - \Phi(w) - \Phi(w') \\
&= m,
\end{aligned}$$

then from Proposition 2, we have $\text{Mod}(\Phi \circ (\text{form}(w), m^*) \circ (\text{form}(w'), m)) = \{w, w'\}$. Hence we have $(\Phi \circ (\text{form}(w), m^*) \circ (\text{form}(w'), m))(w') = (\Phi \circ (\text{form}(w), m^*) \circ (\text{form}(w'), m))(w)$.

From Proposition 1, we have $(\Phi \circ (\text{form}(w), m^*) \circ (\text{form}(w'), m))(w) = (\Phi \circ (\text{form}(w), m^*))(w)$. Recall we already have $(\Phi \circ (\text{form}(w), m^*))(w) = \Phi(w) + m^*$, we get $(\Phi \circ (\text{form}(w), m^*) \circ (\text{form}(w'), m))(w) = \Phi(w) + m^*$, and then we have $(\Phi \circ (\text{form}(w), m^*) \circ (\text{form}(w'), m))(w') = \Phi(w) + m^*$. As $(\Phi \circ (\text{form}(w), m^*) \circ (\text{form}(w'), m))(w') = (\Phi \circ (\text{form}(w'), m) \circ (\text{form}(w), m^*))(w')$, we have $(\Phi \circ (\text{form}(w'), m) \circ (\text{form}(w), m^*))(w') = \Phi(w) + m^*$. Still by Proposition 1, we have $(\Phi \circ (\text{form}(w'), m) \circ (\text{form}(w), m^*))(w') = (\Phi \circ (\text{form}(w'), m))(w')$. So we finally get $(\Phi \circ (\text{form}(w'), m))(w') = \Phi(w) + m^* = \Phi(w) + m + \Phi(w') - \Phi(w) = \Phi(w') + m$.

Step 5, we show that for any w , $(\Phi \circ (\mu, m))(w) = \Phi(w) + m$.

From Steps 1-4, We already have $(\Phi \circ (\text{form}(w), m))(w) = \Phi(w) + m$.

If $w \models \mu$, then from **B4**, we have $(\Phi \circ (\mu, m))(w) = (\Phi \circ (\text{form}(w), m))(w) = \Phi(w) + m$

If $w \models \neg\mu$, then from Proposition 1, we still have $(\Phi \circ (\mu, m))(w) = \Phi(w) = \Phi(w) + (\mu, m)(w)$ as $(\mu, m)(w) = 0$.

So for any w , $(\Phi \circ (\mu, m))(w) = \Phi(w) + m$ holds.

(\Leftarrow) Suppose we have $(\Phi \circ (\mu, m))(w) = \Phi(w) + (\mu, m)(w)$ for any Φ, μ, m and w .

B0 $\Phi \circ (\mu, 0) = \Phi$

For any w , we have $(\Phi \circ (\mu, 0))(w) = \Phi(w) + (\mu, 0)(w) = \Phi(w)$, thus $\Phi \circ (\mu, 0) = \Phi$.

B1 If $f_\Phi(\mu) > m$, then $\Phi \circ (\neg\mu, m) \models \mu$; If $f_\Phi(\mu) < m$, then $\Phi \circ (\neg\mu, m) \models \neg\mu$; If $f_\Phi(\mu) = m$, then $\Phi \circ (\neg\mu, m) \not\models \mu$ and $\Phi \circ (\neg\mu, m) \not\models \neg\mu$.

As for $w \models \mu$, $(\Phi \circ (\neg\mu, m))(w) = \Phi(w)$ and for $w \models \neg\mu$, $(\Phi \circ (\neg\mu, m))(w) = \Phi(w) + m$, we have

$$\begin{aligned} & f_\Phi(\mu) - m \\ = & \Phi(\mu) - \Phi(\neg\mu) - m \\ = & \max_{w \models \mu}(\Phi(w)) - \max_{w \models \neg\mu}(\Phi(w) + m) \\ = & \max_{w \models \mu}((\Phi \circ (\neg\mu, m))(w)) - \max_{w \models \neg\mu}((\Phi \circ (\neg\mu, m))(w)), \end{aligned}$$

so if $f_\Phi(\mu) > m$, then $\max_{w \models \mu}((\Phi \circ (\neg\mu, m))(w)) > \max_{w \models \neg\mu}((\Phi \circ (\neg\mu, m))(w))$, thus $\text{Mod}(\Phi \circ (\neg\mu, m)) = \min(W, \leq_{\Phi \circ (\neg\mu, m)}) \subseteq \text{Mod}(\mu)$ which implies $\Phi \circ (\neg\mu, m) \models \mu$. Similarly, if $f_\Phi(\mu) < m$, we can infer $\Phi \circ (\neg\mu, m) \models \neg\mu$. If $f_\Phi(\mu) = m$, then $\max_{w \models \mu}((\Phi \circ (\neg\mu, m))(w)) = \max_{w \models \neg\mu}((\Phi \circ (\neg\mu, m))(w))$, thus $\min(W, \leq_{\Phi \circ (\neg\mu, m)}) \cap \text{Mod}(\mu) \neq \emptyset$ and $\min(W, \leq_{\Phi \circ (\neg\mu, m)}) \cap \text{Mod}(\neg\mu) \neq \emptyset$ which implies $\Phi \circ (\neg\mu, m) \not\models \mu$ and $\Phi \circ (\neg\mu, m) \not\models \neg\mu$.

B2 If $\Phi \wedge \mu$ is satisfiable, then $\Phi \circ (\mu, m) \equiv \Phi \wedge \mu$.

We only need to show $\text{Mod}(\Phi \wedge \mu) \subseteq \text{Mod}(\Phi \circ (\mu, m))$ and $\text{Mod}(\Phi \circ (\mu, m)) \subseteq \text{Mod}(\Phi \wedge \mu)$.

1. $\text{Mod}(\Phi \wedge \mu) \subseteq \text{Mod}(\Phi \circ (\mu, m))$.

$\forall w \in \text{Mod}(\Phi \wedge \mu)$ and $\forall w'$, we have $w \geq_\Phi w'$ and $(\mu, m)(w) = m \geq (\mu, m)(w')$, thus $\Phi(w) + (\mu, m)(w) \geq \Phi(w') + (\mu, m)(w')$ which is equivalent to $w \geq_{\Phi \circ (\mu, m)} w'$. So $w \in \text{Mod}(\Phi \circ (\mu, m))$.

2. $\text{Mod}(\Phi \circ (\mu, m)) \subseteq \text{Mod}(\Phi \wedge \mu)$.

As $\text{Mod}(\Phi \wedge \mu)$ is satisfiable, let $w^* \in \text{Mod}(\Phi \wedge \mu)$. $\forall w \in \text{Mod}(\Phi \circ (\mu, m))$, it should be $(\Phi \circ (\mu, m))(w) \geq (\Phi \circ (\mu, m))(w^*)$, or $\Phi(w) + (\mu, m)(w) \geq \Phi(w^*) + (\mu, m)(w^*)$. But as $w^* \in \text{Mod}(\Phi \wedge \mu)$, we have $\Phi(w) \leq \Phi(w^*)$ and $(\mu, m)(w) \leq (\mu, m)(w^*)$. Therefore, it should be $\Phi(w) = \Phi(w^*)$ and $(\mu, m)(w) = (\mu, m)(w^*)$ which implies $w \in \text{Mod}(\Phi)$ and $w \models \mu$, so $w \in \text{Mod}(\Phi \wedge \mu)$.

B3 If μ is satisfiable, then $\Phi \circ (\mu, m)$ is satisfiable.

As $(\Phi \circ (\mu, m))(w) = \Phi(w) + m \in Z$, it is obvious.

B4 If $\Phi_1(w_1) = \Phi_2(w_2)$ and $(\mu_1, m)(w_1) = (\mu_2, n)(w_2)$, then $(\Phi_1 \circ (\mu_1, m))(w_1) = (\Phi_2 \circ (\mu_2, n))(w_2)$.

$$\begin{aligned} (\Phi_1 \circ (\mu_1, m))(w_1) &= \Phi_1(w_1) + (\mu_1, m)(w_1) \\ &= \Phi_2(w_2) + (\mu_2, n)(w_2) \\ &= (\Phi_2 \circ (\mu_2, n))(w_2). \end{aligned}$$

B5 $\Phi \circ (\mu, m) \circ (\mu, n) = \Phi \circ (\mu, m + n)$.
 $\forall w$, we have

$$\begin{aligned} (\Phi \circ (\mu, m) \circ (\mu, n))(w) &= (\Phi \circ (\mu, m))(w) + n \\ &= \Phi(w) + m + n \\ &= (\Phi \circ (\mu, m + n))(w). \end{aligned}$$

B6 $\Phi \circ (\mu, m) \circ (\psi, n) = \Phi \circ (\psi, n) \circ (\mu, m)$.
It is easy to obtain that

$$\begin{aligned} (\Phi \circ (\mu, m) \circ (\psi, n))(w) &= (\Phi \circ (\psi, n) \circ (\mu, m))(w) \\ &= \begin{cases} \Phi(w) + m + n & \text{for } w \models \mu \text{ and } w \models \psi, \\ \Phi(w) + m & \text{for } w \models \mu \text{ and } w \not\models \psi, \\ \Phi(w) + n & \text{for } w \not\models \mu \text{ and } w \models \psi, \\ \Phi(w) & \text{for } w \not\models \mu \text{ and } w \not\models \psi, \end{cases} \end{aligned}$$

Proof of Theorem 2: (\Rightarrow) As **B6*** implies **B6**, we have $\forall \Phi, \mu, m, w, (\Phi \circ (\mu, m))(w) = \Phi(w) + (\mu, m)(w)$. Denote $W = \{w_1, \dots, w_n\}$, for any epistemic state Φ , first we show $\Phi = (form(w_1), \Phi(w_1)) \circ \dots \circ (form(w_n), \Phi(w_n))$. In fact, for any w_i , we have $(form(w_i), \Phi(w_i))(w_i) = \Phi(w_i)$ and $\forall j \neq i, (form(w_j), \Phi(w_j))(w_i) = 0$. Hence from Theorem 1, we get $(form(w_1), \Phi(w_1))(w_i) \circ \dots \circ (form(w_n), \Phi(w_n))(w_i) = (form(w_1), \Phi(w_1))(w_i) + \dots + (form(w_n), \Phi(w_n))(w_i) = \Phi(w_i)$. Thus we obtain $\Phi = (form(w_1), \Phi(w_1)) \circ \dots \circ (form(w_n), \Phi(w_n))$. Therefore, for any $w \in W$, we get

$$\begin{aligned} &(\Phi \circ \Phi')(w) \\ &= ((form(w_1), \Phi(w_1)) \circ \dots \circ (form(w_n), \Phi(w_n)) \circ \Phi')(w) \\ &\stackrel{B6*}{=} ((form(w_1), \Phi(w_1)) \circ \dots \circ (form(w_{n-1}), \Phi(w_{n-1})) \circ \Phi' \circ (form(w_n), \Phi(w_n)))(w) \\ &\stackrel{B6*}{=} \dots \\ &\stackrel{B6*}{=} (\Phi' \circ (form(w_1), \Phi(w_1)) \circ \dots \circ (form(w_n), \Phi(w_n)))(w) \\ &= (\Phi' \circ (form(w_1), \Phi(w_1)) \circ \dots \circ (form(w_{n-1}), \Phi(w_{n-1}))(w) \\ &\quad + (form(w_n), \Phi(w_n))(w) \\ &= \dots \\ &= \Phi'(w) + (form(w_1), \Phi(w_1))(w) + \dots + (form(w_n), \Phi(w_n))(w) \\ &= \Phi'(w) + \Phi(w). \end{aligned}$$

So for any w , $(\Phi \circ \Phi')(w) = \Phi(w) + \Phi'(w)$ holds.

(\Leftarrow) Suppose for any w , $(\Phi \circ \Phi')(w) = \Phi(w) + \Phi'(w)$ holds. As postulates **B0-B5** are already proved to be satisfied in Theorem 1, here we only need to show **B6*** holds. This is straightforward and omitted.

Proof of Proposition 3: First, $\forall w \in W$, $\kappa(w) = \max_{w \in W}(\Phi(w)) - \Phi(w) \geq 0$ and there exists some $w' \in \text{Mod}(\Phi)$ that $\kappa(w') = 0$.

Second, $\forall \mu$, we have

$$\begin{aligned} \kappa(\mu) &= \max_{w \in W}(\Phi(w)) - \Phi(\mu) \\ &= \max_{w \in W}(\Phi(w)) - \max_{w \models \mu}(\Phi(w)) \\ &= \max_{w \in W}(\Phi(w)) + \min_{w \models \mu}(-\Phi(w)) \\ &= \min_{w \models \mu}(\max_{w \in W}(\Phi(w)) - \Phi(w)) \\ &= \min_{w \models \mu}(\kappa(w)). \end{aligned}$$

Thus, κ is a valid ordinal conditional function.

Proof of Theorem 3: We have

$$\begin{aligned} &\kappa_{\Phi}(w) + \kappa_{\mu}(w) - \min_{w^* \in W}(\kappa_{\Phi}(w^*) + \kappa_{\mu}(w^*)) \\ = &\max_{w \in W}(\Phi(w)) - \Phi(w) + \max_{w \in W}((\mu, m)(w)) - (\mu, m)(w) \\ &- \min_{w^* \in W}(\max_{w \in W}(\Phi(w)) - \Phi(w^*) + \max_{w \in W}((\mu, m)(w)) - (\mu, m)(w^*)) \\ = &\max_{w \in W}(\Phi(w)) - \Phi(w) + \max_{w \in W}((\mu, m)(w)) - (\mu, m)(w) \\ &- \max_{w \in W}(\Phi(w)) - \max_{w \in W}((\mu, m)(w)) - \min_{w^* \in W}(-\Phi(w^*) - (\mu, m)(w^*)) \\ = &-\Phi(w) - (\mu, m)(w) + \max_{w^* \in W}(\Phi(w^*) + (\mu, m)(w^*)) \\ = &\max_{w \in W}(\Phi(w) + (\mu, m)(w)) - \Phi(w) - (\mu, m)(w) \\ = &\max_{w \in W}((\Phi \circ (\mu, m))(w)) - (\Phi \circ (\mu, m))(w) \\ = &\kappa_{\Phi \circ \mu}(w). \end{aligned}$$

Proof of Theorem 6: First, we define the \preceq_{Φ} as follows

$w_1 \preceq_{\Phi} w_2$ if $\Phi(w_1) \geq \Phi(w_2)$. Conventionally, $w_1 \prec_{\Phi} w_2$ iff $w_1 \preceq_{\Phi} w_2$ and $w_2 \not\preceq_{\Phi} w_1$, and $w_1 =_{\Phi} w_2$ iff $w_1 \preceq_{\Phi} w_2$ and $w_2 \preceq_{\Phi} w_1$.

Obviously, \preceq_{Φ} is total, reflexive, and transitive, thus it is a total pre-order.

Second, we show that the assignment mapping from each Φ to \preceq_{Φ} is faithful:

1. $w_1, w_2 \models \Phi$ only if $w_1 =_{\Phi} w_2$.
 $w_1, w_2 \models \Phi$ implies that $\Phi(w_1) = \Phi(w_2) = \max_{w \in W}(\Phi(w))$, thus we have $w_1 =_{\Phi} w_2$.
2. $w_1 \models \Phi$ and $w_2 \not\models \Phi$ only if $w_1 \prec_{\Phi} w_2$.
 $w_1 \models \Phi$ and $w_2 \not\models \Phi$ implies that $\Phi(w_1) = \max_{w \in W}(\Phi(w))$ and $\Phi(w_2) < \max_{w \in W}(\Phi(w))$, thus $\Phi(w_1) > \Phi(w_2)$, so we have $w_1 \prec_{\Phi} w_2$.
3. $\Phi = \Psi$ only if $\preceq_{\Phi} = \preceq_{\Psi}$.
Follows immediately from the definition of epistemic states and the definition of \preceq_{Φ} and \preceq_{Ψ} .

Third, we show that $Mod(\Phi \circ (\mu, m)) = \min(Mod(\mu), \preceq_\Phi)$ holds.

Follows immediately when μ is not satisfiable. Suppose μ is satisfiable.

1. $Mod(\Phi \circ (\mu, m)) \subseteq \min(Mod(\mu), \preceq_\Phi)$.

Let $\forall w \in Mod(\Phi \circ (\mu, m))$, it should be $w \models \mu$. Otherwise let any $w' \models \mu$, we have $(\Phi \circ (\mu, m))(w) = \Phi(w) + (\mu, m)(w) = \Phi(w) < m^* + \Phi(w') \leq \Phi(w') + (\mu, m)(w')$. Thus $w \notin Mod(\Phi \circ (\mu, m))$ which is a contradiction. So $w \models \mu$. Then as $(\Phi \circ (\mu, m))(w) \geq (\Phi \circ (\mu, m))(w')$, we have $\Phi(w) \geq \Phi(w')$, so $w \preceq_\Phi w'$ for any $w' \models \mu$, hence $w \in \min(Mod(\mu), \preceq_\Phi)$.

2. $\min(Mod(\mu), \preceq_\Phi) \subseteq Mod(\Phi \circ (\mu, m))$.

Let $\forall w \in \min(Mod(\mu), \preceq_\Phi)$, then for any $w' \models \mu$, we have $(\Phi \circ (\mu, m))(w) = \Phi(w) + m \geq \Phi(w') + m = (\Phi \circ (\mu, m))(w')$. And for any $w'' \models \neg\mu$, we have $(\Phi \circ (\mu, m))(w) = \Phi(w) + m > \Phi(w'') = (\Phi \circ (\mu, m))(w'')$. Thus $w \in Mod(\Phi \circ (\mu, m))$.

Proof of Theorem 7:

CR1* If $w_1 \models \mu$ and $w_2 \models \mu$, then $w_1 \preceq_\Phi w_2$ iff $w_1 \preceq_{\Phi \circ (\mu, m)} w_2$.

Follows immediately from $(\Phi \circ (\mu, m))(w) = \Phi(w) + m$ for any $w \models \mu$.

CR2* If $w_1 \models \neg\mu$ and $w_2 \models \neg\mu$, then $w_1 \preceq_\Phi w_2$ iff $w_1 \preceq_{\Phi \circ (\mu, m)} w_2$.

Follows immediately from $(\Phi \circ (\mu, m))(w) = \Phi(w) + m$ for any $w \models \neg\mu$.

CR3* If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 \prec_\Phi w_2$ only if $w_1 \prec_{\Phi \circ (\mu, m)} w_2$.

As $(\Phi \circ (\mu, m))(w_1) = \Phi(w_1) + m > \Phi(w_2)$, it is straightforward.

CR4* If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 \preceq_\Phi w_2$ only if $w_1 \preceq_{\Phi \circ (\mu, m)} w_2$.

As $(\Phi \circ (\mu, m))(w_1) = \Phi(w_1) + m > \Phi(w_2)$, it is straightforward.

Proof of Theorem 8:

RecR* If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 \prec_{\Phi \circ (\mu, m)} w_2$.

As $(\Phi \circ (\mu, m))(w_1) = \Phi(w_1) + m > \Phi(w_2)$, it is straightforward.

IndR* If $w_1 \models \mu$ and $w_2 \models \neg\mu$, then $w_1 \preceq_\Phi w_2$ only if $w_1 \prec_{\Phi \circ (\mu, m)} w_2$.

As $(\Phi \circ (\mu, m))(w_1) = \Phi(w_1) + m > \Phi(w_2)$, it is straightforward.