Recovering Incidence Functions

Weiru Liu   Alan Bundy   Dave Robertson
Dept. of AI,   Univ. of Edinburgh
Edinburgh EH1 1HN

Abstract

In incidence calculus, inferences are usually made by calculating incidence sets and computing probabilities of formulae based on a given incidence function in an incidence calculus theory. Incidence functions are vital for performing any further inference. Without the existence of this function, many of the features of incidence calculus will be lost. However it is still the case that numerical values are assigned on some formulae directly without giving the incidence function. This paper discusses how to recover incidence functions in these cases. The result can be used to calculate mass functions from belief functions in the Dempster-Shafer theory of evidence (or DS theory) and define probability spaces from inner measures (or lower bounds) of probabilities on the relevant propositional language set.

1 Introduction

Incidence calculus [1, 3] as an alternative approach to dealing with uncertainty has a special feature *i.e.,* the indirect association of numerical uncertain assignment on formulae through a set of possible worlds. In this theory, uncertainties are associated with sets of possible worlds and these sets are, in turn, associated with some formulae. This gives incidence calculus the features of both symbolic and numerical reasoning methods. If we take incidence calculus as a symbolic inference technique, it has strong similarity with the ATMS [13]. If we use incidence calculus to make numerical uncertain inference, it can deal with cases for which Dempster-Shafer theory is adequate or inadequate [4, 12]. The crucial point in carrying out the above reasoning procedures relies on a special kind of function, called the incidence function in incidence calculus. Without the existence of this function, many of the features of incidence calculus will be lost. However, in practice numerical values may be required to be assigned on some formulae directly without giving the corresponding incidence function. Therefore it is necessary both theoretically and practically to recover the incidence function in this circumstance. In [2, 3], a preliminary procedure has been described using the Monte Carlo method. This approach has further been developed in [14]. In this paper, we discuss this problem from a different perspective. An alternative approach to defining incidence functions from probability distributions is explored. The result gives a new way to check whether a numerical assignment on a set is a belief function and then calculate its mass functions when it is in DS theory [15, 16] and to construct probability spaces from inner measures (or lower bounds) of probabilities on the relevant propositional language sets [6].

The paper is organized as follows. In section 2, a brief introduction to incidence calculus is given. The key features of incidence functions are discussed. Following this, an algorithm for calculating an incidence function based on numerical assignments is described in section 3. The application of the result to DS theory and probability spaces is described in section 4. Two examples are introduced to show the ideas given in the paper in section 5. Finally a short conclusion is given in section 6.

2 Incidence Calculus

Incidence calculus is a logic for probabilistic reasoning. In incidence calculus, probabilities are not directly associated with formulae, rather sets of possible worlds are directly associated with formulae and probabilities (or lower and upper bounds of probabilities) of formulae are calculated from these sets.

2.1 Generalized Incidence Calculus

In generalized incidence calculus [11][1], a piece of evidence is described in a quintuple called an incidence calculus theory. An incidence calculus theory is normally in the form of $< \mathcal{W}, \varrho, P, \mathcal{A}, i >$ where

- $\mathcal{W}$ is a finite set of possible worlds.
- For all $w \in \mathcal{W}$, $\varrho(w)$ is the probability of w and $wp(\mathcal{W}) = 1$, where $wp(I) = \Sigma_{w \in I} \varrho(w)$.
- P is a finite set of propositions. $\mathcal{A}t$ is the basic element set of P. If P is $\{p_1, ..., p_m\}$, then $\mathcal{A}t$ is defined as for each $\phi \in \mathcal{A}t$, $\phi = \wedge p'_i$ (i=1, ..., m) where $p'_i = p_i$ or $p'_i = \neg p_i$, $\mathcal{L}(P)$ contains all elements produced from P using connectors $\wedge, \vee, \rightarrow, \neg$. For any formula $\phi \in \mathcal{L}(P)$, there exists a subset $A_\phi \in 2^{\mathcal{A}t}$ which makes the following equation hold:

$$\phi = \vee_j \psi_j \quad \psi_j \in A_\phi$$

- $\mathcal{A}$ is a distinguished set of formulae in $\mathcal{L}(P)$ called the *axioms* of the theory.
- i is a function from the axioms $\mathcal{A}$ to $2^{\mathcal{W}}$, the set of subsets of $\mathcal{W}$. $i(\phi)$ is to be thought of as the set of possible worlds in $\mathcal{W}$ in which ϕ is true. $i(\phi)$ is called the *incidence* of ϕ. An incidence function i satisfies the conditions

$$i(\perp) = \{\} \qquad\qquad i(T) = \mathcal{W}$$

Here $\perp$ stands for *False* and T means *True*. For any two formulae ϕ, ψ in $\mathcal{A}$, it is easy to prove that $i(\phi \wedge \psi) = i(\phi) \cap i(\psi)$ if $\phi \wedge \psi$ is in $\mathcal{A}$ based on the definition of i.

[1] The main difference between incidence calculus and generalized incidence calculus is that in generalized incidence calculus there are less conditions on i. See [3, 11] for details.

If we use $\wedge(\mathcal{A})$ to denote the language set which contains $\mathcal{A}$ and all the possible conjunctions of its elements, then this function can be generated to any formula in this set by defining $i(\wedge\phi_j) = \cap_j i(\phi_j)$ if $\wedge_j \phi_j$ is not given initially. Therefore the set of axioms $\mathcal{A}$ can always be extended to a set in which the function i is closed under operator $\wedge$.

Since whenever we have a set of axioms $\mathcal{A}$ with a function i defined on it, where i suits the basic definition of incidences, this set of axioms can always be extended to another set which is closed under the operator $\wedge$ on i. In the following, we always assume that the set of axioms we name is already extended and is closed under $\wedge$, that is $\mathcal{A}$ is closed under $\wedge$. For any two elements in $\mathcal{A}$, we have

$$i(\phi_1 \wedge \phi_2) = i(\phi_1) \cap i(\phi_2) \tag{1}$$

In particular, if $i(\wedge_j \phi_j) = \{\}$, it doesn't matter whether this formula is in $\wedge(\mathcal{A})$ as this formula has no effect on further inferences. However if $\wedge_j \phi_j = \perp$, then $i(\wedge_j \phi_j) = \cap_j i(\phi_j)$ must be empty otherwise the information for constructing the function i is contradictory.

It is not usually possible to infer the incidences of all the formulae in $\mathcal{L}(P)$ given an incidence calculus theory. What we can do is to define both the upper and lower bounds of the incidence using the functions i^* and i_* respectively. For all $\phi \in \mathcal{L}(P)$ these are defined as follows:

$$i^*(\phi) = \mathcal{W} \setminus i_*(\neg\phi) \qquad\qquad i_*(\phi) = \bigcup_{\psi \to \phi = T} i(\psi) \tag{2}$$

where $\psi \to \phi = T$ iff $i(\psi \to \phi) = \mathcal{W}$. For any $\phi \in \mathcal{A}$, we have $i_*(\phi) = i(\phi)$.

The lower bound represents the set of possible worlds in which ϕ is proved to be true and the upper bound represents the set of possible worlds in which $\neg\phi$ fails to be proved. Function $p_*(\phi) = wp(i_*(\phi))$ gives the degree of our belief in ϕ and function $p^*(\phi) = wp(i^*(\phi))$ represents the degree we fail to believe in $\neg\phi$. For a formula ϕ in $\mathcal{A}$, if $p_*(\phi) = p^*(\phi)$, then $p(\phi)$ is defined as $p_*(\phi)$ and is called the probability of this formula.

In the following, when we mention a lower bound of a probability distribution on $\mathcal{A}$, we always mean the function $p_*(*)$ calculated through the lower bound of incidence sets.

2.2 Basic Incidence Assignment

In fact, from an incidence function i, another function ii can be constructed which is called the **basic incidence assignment**. In order to show the relationship between i and ii, we look at an example first. Suppose there are two propositions, $P = \{rainy, windy\}$, and seven possible worlds, $\mathcal{W} = \{sun, mon, tues, wed, thus, fri, sat\}$. Assume that each possible world is equally probable, *i.e.* occurs 1/7 of the time. Through a piece of evidence, we learn that four possible worlds *fri, sat, sun, mon* make *rainy* true, and three possible worlds *mon, wed, fri* make *windy* true. Therefore the incidence sets of these two propositions are:

$$i(rainy) = \{fri, sat, sun, mon\}$$
$$i(windy) = \{mon, wed, fri\}$$

As $i(rainy \wedge windy) = i(rainy) \cap i(windy)$, we also have $i(rainy \wedge windy) = \{fri, mon\}$. So the set of axioms $\mathcal{A}$ is $\mathcal{A} = \{rainy, windy, rainy \wedge windy\}$. The corresponding incidence calculus theory is

$$< \mathcal{W}, \varrho, P, \mathcal{A}, i >$$

and the $\mathcal{A}t$ of P is $\mathcal{A}t = \{rainy \wedge windy, rainy \wedge \neg windy, \neg rainy \wedge windy, \neg rainy \wedge \neg windy\}$. The basic incidence assignment for this theory is

$$ii(rainy \wedge windy) = \{fri, mon\}$$
$$ii(rainy) = \{sat, sun\}$$
$$ii(windy) = \{wed\}$$

It is easy to see that from the basic incidence assignment, the incidence function can be recovered as:

$$i(rainy \wedge windy) = ii(rainy \wedge windy)$$

$$i(rainy) = ii(rainy) \cup ii(rainy \wedge windy)$$

$$i(windy) = ii(windy) \cup ii(rainy \wedge windy)$$

The elements in $ii(\phi)$ make only ϕ true without making any of its superformulae true.

Definition *Basic incidence assignment*

Given a set of axioms $\mathcal{A}$, a function ii defined on $\mathcal{A}$ is called a basic incidence assignment if ii satisfies the following conditions:

$$ii(\phi) \neq \{\} \;\; where \;\; \phi \in \mathcal{A}$$

$$ii(\phi) \cap ii(\psi) = \{\} \;\; where \;\; \phi \neq \psi$$

$$ii(\perp) = \{\} \qquad\qquad ii(T) = \mathcal{W} \setminus \bigcup_j ii(\phi_j)$$

where $\mathcal{W}$ is a set of possible worlds.

Proposition 1 *Given a set of axioms $\mathcal{A}$ with a basic incidence assignment ii, then the function i defined by equation (3) is an incidence function on $\mathcal{A}$.*

$$i(\phi) = \bigcup_{\phi_j \rightarrow \phi = T} ii(\phi_j) \tag{3}$$

PROOF

First of all, because $ii(T) = \mathcal{W} \setminus \cup_j ii(\phi_j)$, we have $i(T) = ii(T) \cup (\cup_j ii(\phi_j)) = \mathcal{W}$. As $ii(\perp) = \{\}$, it is straightforward to infer that $i(\perp) = \{\}$.

Next we are going to prove that $i(\phi \wedge \psi) = i(\phi) \cap i(\psi)$.

Suppose that $i(\phi) \cap i(\psi) = \mathcal{W}' \neq \{\}$,

for each $w \in \mathcal{W}'$, $w \in i(\phi) \cap i(\psi) \Longrightarrow$
$\exists \phi_0$, $w \in ii(\phi_0)$ and $\phi_0 \rightarrow \phi = T$, $\phi_0 \rightarrow \psi = T \Longrightarrow$
$\exists \phi_0, w \in ii(\phi_0)$, and $\phi_0 \rightarrow \phi \wedge \psi = T \Longrightarrow$
$w \in i(\phi \wedge \psi) \Longrightarrow$
$i(\phi) \cap i(\psi) \subseteq i(\phi \wedge \psi)$

Similarly, we can prove that $i(\phi) \cap i(\psi) \supseteq i(\phi \wedge \psi)$, so $i(\phi) \cap i(\psi) = i(\phi \wedge \psi)$. For the case that $i(\phi) \cap i(\psi) = \{\}$, it is still easy to prove that $i(\phi) \cap i(\psi) = i(\phi \wedge \psi)$. Therefore the function i defined by (3) is an incidence function.

QED

Proposition 2 *Given an incidence calculus theory* $< \mathcal{W}, \varrho, P, \mathcal{A}, i >$, *there exists a basic incidence assignment for the incidence function.*

PROOF

This proof procedure is actually to construct a basic incidence assignment ii for the given incidence function.

From the theory $< \mathcal{W}, \varrho, P, \mathcal{A}, i >$, we have

$$i(\phi \wedge \psi) = i(\phi) \cap i(\psi)$$

where $\phi, \psi \in \mathcal{A}$.

The definition of i leads us to the conclusion that if $\psi \rightarrow \phi = T$ then $i(\psi) \subseteq i(\phi)$. As we assume that P is finite, then $\mathcal{A}t, \mathcal{L}(P)$ and $\mathcal{A}$ are all finite.

A subset $\mathcal{A}_0$ of $\mathcal{A}$ can be defined as $\mathcal{A}_0 = \{\psi_1, ..., \psi_n\}$ where $\mathcal{A}_0$ satisfies the condition that

$$\forall \psi_i \in \mathcal{A}_0, \forall \phi \in \mathcal{A},\ \ if\ \ \phi \neq \psi_i\ then\ \ \phi \rightarrow \psi_i \neq T$$

Therefore, $\mathcal{A}_0$ contains the "smallest" formulae in $\mathcal{A}$ and $\mathcal{A}_0$ is not empty. In fact, we can get $\mathcal{A}_0$ using the following procedure. For a formula $\psi_i \in \mathcal{A}$, if $\exists \phi \in \mathcal{A}, \phi \neq \psi_i$ and $\phi \rightarrow \psi_i = T$, then we use ϕ to replace ψ_i and repeat the same procedure until we obtain a formula ϕ_j and we cannot find any formula which makes ϕ_j true, then ϕ_j will be in $\mathcal{A}_0$. For example, the set $\mathcal{A}_0$ in the above example is $\mathcal{A}_0 = \{randy \wedge windy\}$.

For any formula ϕ_i in $\mathcal{A} \setminus \mathcal{A}_0$, there are ψ_{i1}, ..., $\psi_{il} \in \mathcal{A}_0$ where $\psi_{ij} \rightarrow \phi_i = T$. So $i(\psi_{ij}) \subseteq i(\phi_i)$ and $(\bigcup_j i(\psi_{ij})) \subseteq i(\phi_i)$.

Algorithm A

From a function i, we can obtain another function ii using the following procedure:

Step 1: for every formula $\psi \in \mathcal{A}_0$, define $ii(\psi) = i(\psi)$.

Step 2: update $\mathcal{A}$ as $\mathcal{A} \setminus \mathcal{A}_0$.

Step 3: chose a formula ϕ_i in $\mathcal{A}$ which satisfies the requirement that there are $\psi_{i1}, ..., \psi_{il} \in \mathcal{A}_0$ where $\psi_{ij} \to \phi_i = T$ and for any $\phi_j \in \mathcal{A}$, if $\phi_j \neq \phi_i$, then $\phi_j \to \phi_i \neq T$.

Define $ii(\phi_i) = i(\phi_i) \setminus \bigcup_j ii(\psi_{ij})$.

Step 4: delete ϕ_i from $\mathcal{A}$ and update $\mathcal{A}_0$ as $\mathcal{A}_0 \cup \{\phi_i\}$ when $ii(\phi_i) \neq \{\}$. If $\mathcal{A}$ is empty then terminate the procedure. Otherwise go to step 3.

Further defining $ii(T) = \mathcal{W} \setminus \cup_j ii(\phi_j)$, if $ii(T) \neq \{\}$ then $ii(T)$ represents only those possible worlds which make T true. This is also an alternative way to represent ignorance. That is, based on the current information we don't know which formula $ii(T)$ makes true except T. Adding T to $\mathcal{A}_0$, we get a function ii as $ii : \mathcal{A}_0 \to 2^{\mathcal{W}}$. Now we need to prove that ii is a basic incidence assignment. That is, we need to prove

$$ii(\phi_i) \cap ii(\phi_j) = \{\} \qquad where \quad \phi_i \neq \phi_j$$

Suppose that $ii(\phi_i) \cap ii(\phi_j) = W' \neq \{\}$, we have the following inference procedure.

$$\begin{array}{l}
w \in ii(\phi_i) \cap ii(\phi_j) \Longrightarrow \\
w \in i(\phi_i) \quad \text{and} \quad w \in i(\phi_j) \Longrightarrow \\
w \in i(\phi_i) \cap i(\phi_j) \Longrightarrow \\
w \in i(\phi_i \wedge \phi_j) \Longrightarrow \\
\exists \phi \neq \perp \ \wedge \ w \in i(\phi) \ \ and \ \ \phi = \phi_i \wedge \phi_j \Longrightarrow \\
w \notin i(\phi_i) \setminus i(\phi) \ \ or \ \ w \notin i(\phi_j) \setminus i(\phi) \ \ as \ \ \phi_i \neq \phi_j \Longrightarrow \\
w \notin ii(\phi_i) \cap ii(\phi_j)
\end{array}$$

Conflict.

So the equation $ii(\phi_i) \cap ii(\phi_j) = \{\}$ holds for any two distinct elements ϕ_i and ϕ_j in $\mathcal{A}_0$. As we also have $ii(T) = \mathcal{W} \setminus \cup_j ii(\phi_j)$ and $ii(\bot) = i(\bot) = \{\}$, ii is a basic incidence assignment.

QED

3 Recovering an Incidence Function from a Lower Bound of probabilities on a Set of Axioms

Given an incidence calculus theory, we can infer lower bounds of probabilities on formulae. However sometimes numerical assignments are given on some formulae directly without defining any incidence calculus theories. We are interested in how to build incidence calculus theories in these cases. The key part for an incidence calculus theory is to define its incidence function. In this section, we show a way to recover incidence functions in these circumstances.

When we know a proposition set P, its language set $L(P)$, a set of axioms $\mathcal{A}$ and an assignment of lower bound of probabilities on $\mathcal{A}$, our objective is to determine an incidence function i, a set of possible worlds $\mathcal{W}$ and the discrete probability distribution on $\mathcal{W}$ from which the corresponding probability distribution on $\mathcal{A}$ is produced. In order to achieve this goal, we will construct a function ii first and then form i.

For the set of axioms $\mathcal{A}$, we always assume that for $\phi_i, \phi_j \in \mathcal{A}$, $\phi_i \wedge \phi_j \in \mathcal{A}$ and $p(\phi_i \wedge \phi_j)$ is known. If it is not, we will assume that $p(\phi_i \wedge \phi_j) = 0$. When $\phi \rightarrow \phi_i = T$, $i(\phi) \subseteq i(\phi_i)$ and $p(\phi) \leq p(\phi_i)$.

In a similar way as we described in the above section, a special set $\mathcal{A}_0$ is constructible from $\mathcal{A}$ which satisfies the condition

$$\forall \phi \in \mathcal{A}_0, \forall \phi' \in \mathcal{A}, \phi' \rightarrow \phi \neq T, if \;\; \phi \neq \phi' \tag{4}$$

Assume that there are an incidence function i and a basic incidence assignment ii associated with this $\mathcal{A}$, then $w_1 = ii(\phi_i)$ and $w_2 = ii(\phi_j)$

must be two disjoint subsets of an unknown $\mathcal{W}$ because of the feature $ii(\phi_i) \cap ii(\phi_j) = \{\}$ when $\phi_i, \phi_j \in \mathcal{A}_0, \phi_i \neq \phi_j$.

As it is required that the probability distribution on $\mathcal{W}$ should be discrete in incidence calculus, we treat w_1 and w_2 as two single elements in $\mathcal{W}$. The following procedure gives the algorithm for determining the incidence function i, its basic incidence assignment ii and the set of possible worlds with its probability distribution.

Algorithm B

Given $\mathcal{A}$ and a lower bound of probability distribution p_* on $\mathcal{A}$, determine a basic incidence assignment and an incidence function.

Step 1: Assume that $\mathcal{A}_0$ is a subset of $\mathcal{A}$ as defined above in (4).

If there are l elements in $\mathcal{A}_0$, then l elements in $\mathcal{W}$ can be defined from $\mathcal{A}_0$ and define $\varrho(w_i) = p_*(\phi_i)$ for $i = 1, ..., l, \phi_i \in \mathcal{A}_0$.

Further define $ii(\phi_i) = \{w_i\}$, $i(\phi_i) = \{w_i\}$ and $\mathcal{A}' := \mathcal{A} \setminus \mathcal{A}_0$.

Step 2: Chose a formula ψ from $\mathcal{A}'$ which satisfies the condition that $\forall \psi' \in \mathcal{A}'$, $\psi' \rightarrow \psi \neq T$ if $\psi' \neq \psi$.

For all $\phi_j \in \mathcal{A}_0$ repeat $p_*(\psi) := p_*(\psi) - p_*(\phi_j)$ when $\phi_j \rightarrow \psi = T$.

If $p_*(\psi) > 0$ then add an element w_{l+1} to $\mathcal{W}$ and define

$$\begin{aligned}
&ii(\psi) = \{w_{l+1}\} \\
&\varrho(w_{l+1}) = p_*(\psi) \\
&\mathcal{A}_0 := \mathcal{A}_0 \cup \{\psi\} \\
&\mathcal{A}' := \mathcal{A}' \setminus \{\psi\} \\
&i(\psi) = ii(\psi) \cup (\cup_{\phi_j \rightarrow \psi = T} ii(\phi_j)) \\
&l := l + 1
\end{aligned}$$

If $p_*(\psi) = 0$, define $ii(\psi) = \{\}$.

If $p_*(\psi) < 0$, this assignment is not consistent, stop the procedure.

Repeat this step until $\mathcal{A}'$ is empty.

Step 3: Finally if $\Sigma_j(p_*(\phi_j)) < 1$ then add an element w_{l+1} to $\mathcal{W}$ and define

$$\varrho(w_{l+1}) = 1 - \Sigma_j p_*(\phi_j)$$

$$ii(T) = \{w_{l+1}\}$$

Step 4: The resulting set of possible worlds is $\mathcal{W} = \{w_1, w_2, ..., w_{l+1}\}$ and the probability distribution is $\varrho(w_i) = p_*(\phi_i)$ where $\phi_i \in \mathcal{A}_0$ and $\Sigma_i \varrho(w_i) = 1$. Two functions ii and i are defined as $ii(\phi_i) = \{w_i\}$ and $i(\phi) = \cup_{\phi_j \to \phi} ii(\phi_j)$, $\phi_j \in \mathcal{A}_0$.

It is easy to prove that ii and i are a basic incidence assignment and an incidence function respectively. The corresponding incidence calculus theory is $< \mathcal{W}, \varrho, P, \mathcal{A}, i >$.

If there are n elements in $\mathcal{A}$ then there are at most $n+1$ elements in $\mathcal{W}$.

This algorithm is entirely based on the result that $ii(\phi) \cap ii(\psi) = \{\}$. In algorithm B, for a formula ϕ, we keep deleting those portions in $p_*(\phi)$ which can be carried by its superformulae until we obtain the last bit which must be carried by ϕ itself. Then the last portion will only be contributed by its basic incidence set.

4 Extending the Result to DS Theory and Probability Spaces

One of the meaningful extensions of this algorithm is to calculate the mass function in DS theory when $\mathcal{A}$ is the whole language set $\mathcal{L}(P)$ and p_* is a belief function on it [15, 16] and, in particular, to recover the corresponding probability space when p_* is thought of as an inner measure (or a lower bound) on $\mathcal{A}$ in probability structures [6].

One may suspect that *bel* is usually defined on a frame of discernment[2] in DS theory rather on a set of formulae. We will briefly show how to

[2] A set is defined as a frame of discernment if this set contains mutually exclusive and exhaustive answers for a question.

build a belief function on a set of formulae here, more details can be found in [6].

Assume that we have a set of propositions P and its basic element set $\mathcal{A}t$. Because $\mathcal{A}t$ satisfies the definition of a frame of discernment, we can talk about a belief function on $\mathcal{A}t$. Further if we follow the one-to-one relationship between $2^{\mathcal{A}t}$ and $\mathcal{L}(P)$ as we have seen in section 2, then given a belief function bel on $\mathcal{A}t$, we can define a belief function on $\mathcal{L}(P)$ as $bel'(\phi) = bel(A_\phi)$ where $A_\phi \subseteq \mathcal{A}t$. Therefore we can also talk about a belief function on a language set $\mathcal{L}(P)$.

4.1 Calculating mass functions

In DS theory, a function on a frame Θ is called a mass function, denoted as m if $\Sigma_A m(A) = 1$ where $A \subseteq \Theta$. The relationship between a belief function, denoted as bel, and its mass function is unique. They can be recovered from each other as follows.

$$bel(A) = \Sigma_{B \subseteq A} m(B)$$
$$m(A) = \Sigma_{B \subseteq A, B \neq \emptyset} (-1)^{a-b} bel(B)$$

where $a - b = \mid (A \wedge \neg B) \mid$ where $A, B \in L(P)$ [16]. $\mid A \mid$ stands for the element number in A.

In the following we show an alternative way to obtain a mass function from a belief function by means of incidence calculus. Assume that $\mathcal{A}$ is the whole language set $\mathcal{L}(P)$ and p_* is a belief function on $\mathcal{A}$, then p_* is also a lower bound of probability on $\mathcal{A}$ in incidence calculus as shown in [4, 12].

Algorithm C

Given a function bel on the set $\mathcal{L}(P) = \mathcal{A}$, determine whether bel is a belief function on this language set [3] and obtain its mass function if it is.

Step 1: Delete all those elements in $\mathcal{A}$ in which $bel(*) = 0$. Then as in algorithm B, define a subset $\mathcal{A}_0$ out of $\mathcal{A}$. For any $\phi \in \mathcal{A}_0$, define

[3] In fact, this language set can be any frame of discernment.

$m(\phi) = bel(\phi)$. Assume that there are l elements in $\mathcal{A}_0$. Define $\mathcal{A}' = \mathcal{A} \setminus \mathcal{A}_0$.

Step 2: Chose a formula ψ from $\mathcal{A}'$ which satisfies the condition that $\forall \psi' \in \mathcal{A}'$, $\psi' \rightarrow \psi \neq T$.

For all $\phi_j \in \mathcal{A}_0$ repeat $bel(\psi) := bel(\psi) - bel(\phi_j)$ when $\phi_j \rightarrow \psi = T$.

If $bel(\psi) > 0$, define

$$
\begin{aligned}
& l := l + 1 \\
& \mathcal{A}_0 := \mathcal{A}_0 \cup \{\psi\} \\
& \mathcal{A}' := \mathcal{A}' \setminus \{\psi\} \\
& m(\psi) := bel(\psi)
\end{aligned}
$$

If $bel(\psi) = 0$ then ψ is not a focal element[4] of this belief function.

If $bel(\phi) < 0$ then this assignment is not a belief function, stop the procedure.

Repeat this step until $\mathcal{A}'$ is empty.

Step 3: All the elements in $\mathcal{A}_0$ will be the focal elements of this belief function and the function m defined in Step 2 is the corresponding mass function. It is easy to prove that $\Sigma_A m(A) = 1$.

The algorithm tries to find the focal elements of a belief function one by one. Once all the focal elements are fixed and the uncertain values of these elements are defined, the corresponding mass function is known. The worst case of computational complexity of this algorithm is the same as the approach used in DS theory but it may be more efficient when the elements in $\mathcal{A}'$ are arranged in the decreasing sequence of their sizes. However the Fast Moebius Transform of Kennes and Smets remains faster than ours [7, 8, 9].

[4]When $m(A) > 0$, A is called a focal element of its belief function.

4.2 Recovering probability spaces

In [5, 6], given a probability space $(\mathcal{W}, \chi, \varrho)$, an inner measure on a propositional language set can be defined through a mapping $\pi(w) : \mathcal{L}(P) \rightarrow \{\mathbf{true}, \mathbf{false}\}$. If $\pi(w)(\phi) = \mathbf{true}$, ϕ is said to be true at w; otherwise we say that ϕ is false at w. ϕ^{π} is defined to contain all those elements in $\mathcal{W}$ in which ϕ is true. If we define $\mu_*(\phi) = \varrho_*(\phi^{\pi})$, then μ_* is called an inner measure of a probability on $\mathcal{L}(P)$. It is proved in [6] that a belief function on such a language set is also an inner measure on this set which is generated from a probability space. Therefore it is also interesting to apply the above technique to recover a probability space when we know an inner measure μ_* (or lower bound) of probabilities on a language set.

Following the Algorithm C, in Step 2 when $bel(\phi) > 0$ if we further assign

$$ii(\phi) = W_j \qquad \varrho(W_j) = bel(\phi)$$

where W_j is a subset of a set $\mathcal{W}$, then for any two elements in $\mathcal{A}_0$ we have

$$ii(\phi_i) \cap ii(\phi_j) = \{\}$$

That is $W_i \cap W_j = \{\}$. Therefore W_i, $i = 1, ..., n$ are disjoint subsets of $\mathcal{W}$ and $\Sigma_i \varrho(W_i) = 1$. So $\chi' = \{W_1, W_2, ..., W_n\}$ can be a basis for a σ-algebra. The corresponding probability space will be $(\mathcal{W}, \chi, \varrho)$ where χ is the $\sigma-$algebra generated by the basis χ'. This mapping π can basically be defined as $\pi(w_{ij})(\phi_i) = \mathbf{true}$, $w_{ij} \in ii(\phi_i) = W_i$. Therefore the corresponding probability structure is $(\mathcal{W}, \chi, \varrho, \pi)$. From this structure, the given μ_* can be recovered. More details about probability space, probability structure and its relation with DS theory can be found in [5, 6].

5 Examples

In this section, we use two examples to show our algorithms in this paper. The first example is reconstructed from [10].

Example 1:

Assume that we know the probability distribution on a set of axioms of formulae, we want to create a set of possible worlds and its probability distribution and to define an incidence function from this set to the set of axioms. The created set of possible worlds and the incidence function can, in turn, produce the probability distribution on the set of axioms.

Suppose that we have P, $\mathcal{L}(P)$ and a set of axioms $\mathcal{A}$ as $\mathcal{A} = \{a, b, c, a \wedge b, a \wedge c, b \wedge c, a \wedge b \wedge c\}$ with a lower bound of a probability distribution as

$$
\begin{array}{ll}
p_*(a) = 0.760 & p_*(b) = 0.640 \\
p_*(c) = 0.480 & p_*(a \wedge b) = 0.525 \\
p_*(a \wedge c) = 0.350 & p_*(b \wedge c) = 0.225 \\
p_*(a \wedge b \wedge c) = 0.165 &
\end{array}
$$

The set $\mathcal{A}$ is closed under operator $\wedge$. Following Algorithm B, an incidence function is defined by the following steps.

Step 1. The set $\mathcal{A}_0$ is $\{a \wedge b \wedge c\}$ which contains the smallest formula in $\mathcal{A}$. So there is at least one possible world w_1 supposing formula $a \wedge b \wedge c$ and $\varrho(w_1) = 0.165$. We also have

$$
\begin{aligned}
& i(a \wedge b \wedge c) = ii(a \wedge b \wedge c) = \{w_1\} \\
& \mathcal{A}' = \mathcal{A} \setminus \mathcal{A}_0 \\
& l := 1
\end{aligned}
$$

Step 2. Chose a formula $a \wedge b$ from $\mathcal{A}'$, as only formula $a \wedge b \wedge c$ possesses the feature that $a \wedge b \wedge c \rightarrow a \wedge b = T$, We have

$$p_*(a \wedge b) := p_*(a \wedge b) - p_*(a \wedge b \wedge c) = 0.525 - 0.165 = 0.36$$

Because $p_*(a \wedge b) > 0$, we define

$$
\begin{aligned}
& ii(a \wedge b) = \{w_2\} \\
& \varrho(w_2) = p_*(a \wedge b) \\
& \mathcal{A}_0 := \mathcal{A}_0 \cup \{a \wedge b\}
\end{aligned}
$$

$$\mathcal{A}' := \mathcal{A}' \setminus \{a \wedge b\}$$
$$i(a \wedge b) = \{w_1, w_2\}$$
$$l := l + 1$$

Repeat this step for all the elements in $\mathcal{A}'$, we get

$$\begin{array}{lll} ii(a \wedge c) = \{w_3\} & \varrho(w_3) = 0.185 & i(a \wedge c) = \{w_1, w_3\} \\ ii(b \wedge c) = \{w_4\} & \varrho(w_4) = 0.06 & i(b \wedge c) = \{w_1, w_4\} \\ ii(a) = \{w_5\} & \varrho(w_5) = 0.05 & i(a) = \{w_1, w_2, w_3, w_5\} \\ ii(b) = \{w_6\} & \varrho(w_6) = 0.055 & i(b) = \{w_1, w_2, w_4, w_6\} \\ ii(c) = \{w_7\} & \varrho(w_7) = 0.070 & i(c) = \{w_1, w_3, w_4, w_7\} \end{array}$$

Eventually, define $wp(ii(T)) = 1 - \Sigma_j wp(ii(\phi_j)) = 1 - wp(\{w_1, ..., w_7\}) = 0.055$ and let $ii(T) = \{w_8\}$, then we obtain $\mathcal{W} = \{w_1, ..., w_8\}$ with probability distribution ϱ on it.

For any other formula ψ, if $wp(ii(\psi)) = 0$, we explain this in two ways: there is no any possible world making this formula true or the probability of the subset which makes ψ true is 0. In any case, it doesn't matter whether we add $ii(\psi)$ to the whole set of possible worlds or not. The incidence calculus theory which can produce the probability distribution p on $\mathcal{A}$ is $< \mathcal{W}, \varrho, P, \mathcal{A}, i >$.

For any formula $\phi \in \mathcal{L}(P) \setminus \mathcal{A}$, we can calculate both $i_*(\phi)$ and $p_*(\phi)$.

Example 2:

Assume that there are four elements in $\mathcal{A}t = \{a, b, c, d\}$ and $\mathcal{A} = \mathcal{L}(P)$ is $\mathcal{A} = \{a, b, c, d, a \vee b, a \vee c, a \vee d, b \vee c, b \vee d, c \vee d, a \vee b \vee c, a \vee c \vee d, a \vee b \vee d, b \vee c \vee d, a \vee b \vee c \vee d\}$ and the corresponding degrees of belief in elements of $\mathcal{A}$ are $bel(\mathcal{A}) = \{.5, 0, 0, .3, .7, .5, 8, 0, .3, .3, .7, 8, 1, .3, 1\}$.

By using the Algorithm C, the calculating procedure for a mass function is as follows.

Step 1. After deleting those elements with 0 degrees of belief, we have $\mathcal{A} = \{a \vee b \vee c \vee d, b \vee c \vee d, a \vee b \vee d, a \vee c \vee d, a \vee b \vee d, c \vee d, b \vee d, a \vee d, a \vee c, a \vee b, d, a\}$ and $\mathcal{A}_0 = \{a, d\}$. Define $m(a) = bel(a) = .5$, $m(d) = bel(d) = .3$, $l = 2$ and $\mathcal{A}' := \mathcal{A} \setminus \mathcal{A}_0$.

Step 2. Get $a \vee c$ from $\mathcal{A}'$. Because $a \rightarrow a \vee c = T$, we have $bel(a \vee c) := bel(a \vee c) - bel(a) = .5 - .5 = 0$. So $a \vee c$ is not a focal element. Repeat this procedure until we get $a \vee b$ and we have $bel(a \vee b) := .7 - .5 = .2$. Define

$$
\begin{aligned}
& m(a \vee b) = bel(a \vee b) = .2 \\
& \mathcal{A}_0 := \mathcal{A}_0 \cup \{a \vee c\} \\
& \mathcal{A}' := \mathcal{A}' \setminus \{a \vee c\} \\
& l := l + 1
\end{aligned}
$$

Repeat this procedure until $\mathcal{A}'$ is empty, we get $\mathcal{A}_0 = \{a, d, a \vee b\}$ and the mass function m is $m(a) = .5, m(d) = .3, m(a \vee c) = .2$.

If we take bel as an inner measure of a probability on $\mathcal{A}$ from an unknown probability space, this space can be recovered as $(\mathcal{W}, \chi, \varrho)$ where the basis for χ is $\chi' = \{W_1, W_2, W_3\}$, $W_1 \cup W_2 \cup W_3 = \mathcal{W}$ and $\varrho(W_1) = .5, \varrho(W_2) = .3, \varrho(W_3) = .2$.

The computational complexity may be high when there are huge number of elements in $\mathcal{L}(P)$. That is, it is exponential with the element number of P.

6 Summary

Dealing with uncertainty is an important task in many automated reasoning systems. Quite a few numerical and symbolic approaches have been proposed and discussed. In this paper, we focused on incidence calculus and mainly on the recovering procedure from numerical assignments to symbolic assignments. The result shows that numerical assignments and symbolic assignments can be transformed into each other in some circumstances.

We have discussed an approach to defining an incidence function based on a probability measure in incidence calculus. The advantage of this approach is that its computational complexity is lower *i.e.* o($\mid A \mid$) comparing to the method discussed in [14]. The latter is exponential given the same set of axioms $\mathcal{A}$. The size of the set of possible worlds entirely depends on the size of $\mathcal{A}$. For example, if there are only two elements in $\mathcal{A}$, then we can define a set of possible worlds containing at most three elements. This is mainly because the probability distribution on the set of possible worlds must be discrete.

When we extend the result to DS theory and the probability space, we follow the known result that a lower bound in incidence calculus is equivalent to a belief function and a belief function is, in turn, equivalent to an inner measure in probability structures when these three theories concern the same problem space. Therefore the incidence assignment procedure can be not only used to define an incidence assignment but also used to construct an undefined probability space. In the latter case, a basis for an $\sigma-$algebra of a probability space is similar to a set of possible worlds except that each subset in the basis usually contains more than one elements.

Acknowledgement

The authors are grateful for comments from Prof. P.Smets.

The first author is supported by a "Colin and Ethel Gordon Scholarship" of the Faculty of Science and Engineering, Edinburgh University and an ORS award.

References

[1] Bundy,A., Incidence calculus: a mechanism for probability reasoning, *J. of Automated Reasoning.* 1, 263-283, 1985.

[2] Bundy,A., Correctness criteria of some algorithms for uncertain reasoning using incidence calculus., *J. of Automated reasoning.* 2 109-126., 1986.

[3] Bundy,A., Incidence Calculus, *The Encyclopedia of AI*, 663-668, 1992. It is also available as the Research paper No. 497 in the Dept. of Artificial Intelligence, Univ. of Edinburgh.

[4] Correa da Silva,F. and A.Bundy, On some equivalent relations between incidence calculus and Dempster-Shafer theory of evidence, *Proc. of sixth workshop of Uncertainty in Artificial Intelligence.* 378-383, 1990.

[5] Dempster,A.P., Upper and lower probabilities induced by a multivalued mapping, *Ann. Math. Stat.*. 38, 325-339, 1967.

[6] Fagin,R. and J. Halpern, Uncertainty, belief and probability, Research Report of IBM, RJ 6191, 1989.

[7] Kennes,R. Computational aspects of the Moebius transform of a graph, *IEEE-SMC,* 22:201-223, 1991.

[8] Kennes,R. and Smets,Ph., Computational aspects of the Moebius Transform. *Proc. of the 6th Conf. on Uncertainty in AI* Eds. by P.Bonissone, M.Henrion, L.Kanal and J.Lemmer. Cambridge, MA. North Holland, 401-416, 1990a.

[9] Kennes,R. and Smets,Ph., Fast algorithms for Dempster-Shafer theory. *Proc. of the 3rd Inter. Conf. on Information Processing and Management of Uncertainty*, Paris, France, 2-6, 1990b. (Full version to appear in Lecture Notes in Computer Science Series Eds. by B.Bouchon-Meunier, R.R.Yager, L.A.Zadeh, Springer Verlag.)

[10] H.K.Kyburg Jr., Evidential probability, Computer Science Technical Report number 376, Univ. of Rochester.

[11] Liu,W., Extended incidence calculus. Chapter 3 of a forthcoming PhD thesis. Dept. of AI, Univ. of Edinburgh, 1993.

[12] Liu,W. and A.Bundy, The combination of different pieces of evidence using incidence calculus, Research Paper 599, Dept. of Artificial Intelligence, Univ. of Edinburgh, 1992.

[13] Liu,W., A.Bundy and D.Robertson, On the relationship between incidence calculus and the ATMS, in this proceedings, 1993.

[14] McLean,R.G., *Testing and Extending the Incidence Calculus*, M.Sc. Dissertation, Dept. of Artificial Intelligence, Univ. of Edinburgh, 1992.

[15] Shafer,G., *A mathematical theory of evidence*, Princeton University Press. 1976.

[16] Smets,P., Belief functions, *Non-Standard Logics for Automated Reasoning*, (Smets, Mamdani, Dubois and Prade Eds.), 253-286, 1988.